# An $H^1$ setting for the Navier-Stokes equations: quantitative estimates

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#### **Abstract**

We consider the incompressible Navier-Stokes (NS) equations on a torus, in the setting of the spaces  $L^2$  and  $H^1$ ; our approach is based on a general framework for semi- or quasi-linear parabolic equations proposed in the previous work [9]. We present some estimates on the linear semigroup generated by the Laplacian and on the quadratic NS nonlinearity; these are fully quantitative, i.e., all the constants appearing therein are given explicitly. As an application we show that, on a three dimensional torus  $\mathbf{T}^3$ , the (mild) solution of the NS Cauchy problem is global for each  $H^1$  initial datum  $u_0$  with zero mean, such that  $||\operatorname{curl} u_0||_{L^2} \leq 0.407$ ; this improves the bound for global existence  $||\operatorname{curl} u_0||_{L^2} \leq 0.00724$ , derived recently by Robinson and Sadowski [10]. We announce some future applications, based again on the  $H^1$  framework and on the general scheme of [9].

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### 1 Introduction

As well known, the great open problem about the incompressible Navier-Stokes (NS) equations is to prove global existence of the solutions in three space dimensions, for all sufficiently smooth initial data.

A much more modest, but realistic research program about these equations is the following.

- (i) Deriving estimates for the linear semigroup generated by the Laplacian and for the NS quadratic nonlinearity, within an appropriate functional setting (say, the Sobolev spaces on  $\mathbb{R}^3$  or on the torus  $\mathbb{T}^3$ ). This should be done paying attention to the strictly quantitative aspects, and having as a goal the best achievable accuracy.
- (ii) Explicitating the estimates for the NS solutions derivable from the previous framework.

Some typical issues to be treated as parts of item (ii), always using the information from (i), are the following ones:

- (ii.a) Deriving quantitative lower bounds on the interval of existence of the solution, for a given initial datum.
- (ii.b) Producing a sufficient condition for global existence of the solution, when the norm of the initial datum is below an *explicitly given* upper bound.
- (ii.c) Devising a posteriori tests on the approximate solutions (e.g., of Galerkin type) to get quantitative estimates on exact solutions.

A number of papers published in recent times are more or less related to the above research program. We mention, in particular: the works by Chernyshenko, Constantin, Robinson, Titi [1], Dashti and Robinson [2] on the approximate solutions of the NS equations, and on the a posteriori estimates derivable from them; the paper by Robinson and Sadowski [10] on the conditions for the existence of (exact) global solutions, containing very interesting considerations on the computational times required to check them via suitable approximate solutions; our papers [7] [9] on approximate solutions and a posteriori estimates for general semi- or quasi-linear evolution equations, including (in the second reference) some applications to NS equations on a torus. Let us also mention that a similar attitude towards approximate solutions and a posteriori bounds has been developed by Machiels, Peraire and Patera [6], Veroy and Patera [12] in connection with the NS equations (or some related space discretizations), dealing mainly with two-dimensional steady problems.

The present work outlines a framework to treat problems (ii.a)(ii.b)(ii.c) for the incompressible NS equations on the torus  $\mathbf{T}^3$ . The main application presented here is a fully quantitative bound on the initial datum yielding global existence, in the spirit of (ii.b); a few words will be spent on other issues within the scheme of (ii), to be treated elsewhere.

Our present constructions are related to the setup of [9], where the basic functional

space for the three dimensional NS equations was the Sobolev space  $\mathbb{H}^n_{\Sigma_0}(\mathbf{T}^3) \subset \mathbb{H}^n_0(\mathbf{T}^3)$ , with n > 3/2 (the subscripts  $\Sigma$ , 0 indicating the conditions of zero divergence and zero mean, respectively).

On the contrary, here the solutions of the NS equations take values in  $\mathbb{H}^1_{\Sigma_0}(\mathbf{T}^3) \subset \mathbb{H}^1_0(\mathbf{T}^3)$ .  $\mathbb{H}^1_0(\mathbf{T}^3)$  is equipped with the norm  $v \mapsto ||v||_1 := ||\sqrt{-\Delta} v||_{L^2}$  ( $\Delta$  the Laplacian); the latter is restricted to  $\mathbb{H}^1_{\Sigma_0}(\mathbf{T}^3)$ , where it is found to coincide with the  $L^2$  norm of the vorticity:

$$||v||_1 = ||\operatorname{curl} v||_{L^2} . {1.1}$$

Throughout the paper, the incompressible NS equations for the velocity field u are written as

$$\frac{du}{dt} = \Delta u - \mathfrak{L}(u \cdot \partial u) , \qquad (1.2)$$

with  $\mathfrak{L}$  indicating the Leray projection on the divergence free vector fields. As in our previous work, we emphasize the role played by the constants in certain basic estimates on the semigroup  $(e^{t\Delta})$  and on the NS bilinear map  $(v, w) \mapsto \mathfrak{L}(v \cdot \partial w)$ . In particular, for  $1/2 < \omega < 1$  we consider the negative order Sobolev space  $\mathbb{H}_{\Sigma_0}^{-\omega}(\mathbf{T}^3)$  with the norm  $\|\cdot\|_{-\omega} := \|\sqrt{\Delta}^{-\omega}\cdot\|_{L^2}$ , and give explicitly a constant  $K_{\omega}$  such that, for all  $v, w \in \mathbb{H}_{\Sigma_0}^1(\mathbf{T}^3)$ ,

$$\|\mathfrak{L}(v \bullet \partial w)\|_{-\omega} \leqslant K_{\omega} \|v\|_1 \|w\|_1 ; \qquad (1.3)$$

for example, if  $\omega = 7/10$  the above inequality is fulfilled with  $K_{7/10} = 0.361$ . On the grounds of our previous investigations on multiplication in Sobolev spaces [8], the constant  $K_{\omega}$  determined for any  $\omega$  by our method is probably close to the best constant fulfilling (1.3).

The evaluation of  $K_{\omega}$  can be combined with some strictly quantitative estimates on  $(e^{t\Delta})$ , and with the general setting of [7], to obtain a number of results on the exact NS solutions. In particular, here we show that the incompressible NS equations have a global solution u for any initial datum  $u_0 \in \mathbb{H}^1_{\Sigma_0}(\mathbf{T}^3)$  such that

$$||u_0||_1 \leqslant 0.407 \; ; \tag{1.4}$$

this improves the quantitative condition for global existence presented in [10] (page 43) that, in our notations, can be written as  $||u_0||_1 \leq 0.00724$ . In connection with these results, let us mention that the idea of an explicit norm bound, sufficient for global existence, dates back to the seminal paper by Kato and Fujita [3]; here, an approach slightly different from ours was outlined but not developed in a fully quantitative way, since the authors did not compute all the necessary constants (nor indicated how to do the missing calculations).

Let us outline the organization of the present paper. In Section 2, we review some part of the abstract framework of [9] for semi- or quasi-linear evolution equations; we

consider in particular the case of a quadratic nonlinearity, suitable for application to the NS equations.

In Section 3, we fix our standards about Sobolev spaces on a torus  $\mathbf{T}^d$  (for the moment, of any dimension  $d \geq 2$ ), and indicate the connections of this formalism with the NS equations (of course a precise definition of these standards, and especially of the Sobolev norms, is required to make meaningful the quantitative estimates mentioned before, to have comparisons with [10] and other papers, etc.).

In Section 4 we discuss the semigroup  $(t, v) \mapsto e^{t\Delta}v$  as a map  $[0, +\infty) \times \mathbb{H}^1_{\Sigma_0}(\mathbf{T}^d) \to \mathbb{H}^1_{\Sigma_0}(\mathbf{T}^d)$ , or as a map  $(0, +\infty) \times \mathbb{H}^{-\omega}_{\Sigma_0}(\mathbf{T}^d) \to \mathbb{H}^1_{\Sigma_0}(\mathbf{T}^d)$ , for  $d \geq 2$  and  $0 < \omega < 1$ . In Section 5, after some preliminaries we discuss the bilinear map  $\mathbb{H}^1_{\Sigma_0}(\mathbf{T}^d) \times \mathbb{H}^1_{\Sigma_0}(\mathbf{T}^d) \to \mathbb{H}^{-\omega}_{\Sigma_0}(\mathbf{T}^d)$ ,  $(v, w) \mapsto \mathfrak{L}(v \cdot \partial w)$  for  $d \geq 2$  and  $\omega > d/2 - 1$ ; we infer an inequality of the type (1.3) and show how to compute the related constant  $K_{\omega}$  (in fact, also dependent on the dimension d).

In Section 6, we put together the results of Sections 4-5 on  $(e^{t\Delta})$  and on the NS bilinear map, which is possible if  $(d=2, 0 < \omega < 1 \text{ or})$   $d=3, 1/2 < \omega < 1$ ; this gives our final " $H^1$  framework" for the NS equations on  $\mathbf{T}^3$ . As an output of this framework, choosing  $\omega = 7/10$  we obtain the condition (1.4) of global existence in  $\mathbb{H}^1_{\Sigma 0}(\mathbf{T}^3)$ , together with a quantitative estimate on the exponential decay rate of the global solution u.

In Section 7 we present a method to get estimates on the (exact) NS solution analyzing a posteriori any approximate solution, in the  $H^1$  framework on  $\mathbf{T}^3$ . Here we specialize the general method of the "control inequality" introduced in [7] [9]; this was already applied in [9] to the NS equations on  $\mathbb{H}^n_{\Sigma_0}(\mathbf{T}^3)$ , n > 3/2, for example to get estimates from the Galerkin approximate solutions. Some applications of the control inequality to the approximate NS solutions in  $\mathbb{H}^1_{\Sigma_0}(\mathbf{T}^3)$  will be presented elsewhere.

The paper is completed by some Appendices, concerning more technical issues. In Appendix A, we prove some estimates on the semigroup  $(e^{t\Delta})$  employed in Sections 3-4. In Appendix B we illustrate some general facts on (discrete) convolutions of unimodal functions; these results are employed in the subsequent Appendix C, where we study a convolution whose maximum gives the constant  $K_{\omega}$  of Eq. (1.3). In Appendix D we give some details related to the computation of  $K_{\omega}$  in dimension d=3, for  $\omega=7/10$ .

## 2 An abstract framework for evolution equations with a quadratic nonlinearity

In [9], we considered a framework for evolutionary problems (mainly, of parabolic type) with a nonlinear part of a fairly general kind; in the cited paper, the framework was subsequently specialized to the case of a quadratic nonlinearity. For our present

purposes, it suffices to review the case of a quadratic, time independent nonlinearity. **The framework.** Throughout this section, we consider a set

$$(\mathbf{F}_{+}, \mathbf{F}, \mathbf{F}_{-}, \mathcal{A}, \mathcal{P}) \tag{2.1}$$

with the following properties.

(P1)  $\mathbf{F}_+$ ,  $\mathbf{F}$  and  $\mathbf{F}_-$  are Banach spaces with norms  $\| \cdot \|_+$ ,  $\| \cdot \|_-$  and  $\| \cdot \|_-$ , such that

$$\mathbf{F}_{\perp} \hookrightarrow \mathbf{F} \hookrightarrow \mathbf{F}_{\perp}$$
 (2.2)

(the symbol  $\hookrightarrow$  indicating that one space is a dense linear subspace of the other, and that the natural inclusion is continuous). Elements of these spaces are generically denoted with  $v, w, \dots$ .

(P2)  $\mathcal{A}$  is a linear operator such that

$$\mathcal{A}: \mathbf{F}_{+} \to \mathbf{F}_{-} , \qquad v \mapsto \mathcal{A}v .$$
 (2.3)

Viewing  $\mathbf{F}_+$  as a subspace of  $\mathbf{F}_-$ , the norm  $\| \cdot \|_+$  is equivalent to the graph norm  $v \in \mathbf{F}_+ \mapsto \|v\|_- + \|Av\|_-$ .

(P3) Viewing  $\mathcal{A}$  as a densely defined linear operator in  $\mathbf{F}_{-}$ , it is assumed that  $\mathcal{A}$  generates a strongly continuous semigroup  $(e^{t\mathcal{A}})_{t\in[0,+\infty)}$  on  $\mathbf{F}_{-}$  (of course, from the standard theory of linear semigroups, we have  $e^{t\mathcal{A}}(\mathbf{F}_{+})\subset\mathbf{F}_{+}$  for all  $t\geqslant 0$ ).

(P4) One has

$$e^{t\mathcal{A}}(\mathbf{F}) \subset \mathbf{F} \quad \text{for } t \in [0, +\infty) ;$$
 (2.4)

the function  $(v,t) \mapsto e^{tA}v$  is continuous from  $\mathbf{F} \times [0,+\infty)$  to  $\mathbf{F}$ , yielding a strongly continuous semigroup on  $\mathbf{F}$  as well. Furthermore, there is a constant B > 0 such that

$$||e^{t\mathcal{A}}v|| \leqslant e^{-Bt}||v|| \quad \text{for all } v \in \mathbf{F} .$$
 (2.5)

(P5) One has

$$e^{t\mathcal{A}}(\mathbf{F}_{-}) \subset \mathbf{F} \quad \text{for } t \in (0, +\infty) ;$$
 (2.6)

the function  $(v,t) \mapsto e^{tA}v$  is continuous from  $\mathbf{F}_- \times (0,+\infty)$  to  $\mathbf{F}$  (in a few words: for all t > 0,  $e^{tA}$  regularizes the vectors of  $\mathbf{F}_-$ , sending them into  $\mathbf{F}$  continuously). Furthermore, there is a function  $\mu \in C((0,+\infty),(0,+\infty))$  such that

$$||e^{tA}v|| \le \mu(t) e^{-Bt} ||v||_{-}$$
 for  $t > 0, v \in \mathbf{F}_{-}, B$  as in (2.5). (2.7)

The function  $\mu$  behaves like an integrable power of 1/t for  $t \to 0$ , i.e.,

$$\mu(t) = O(1/t^{\varepsilon}) \quad \text{for } t \to 0^+ , \quad \varepsilon \in [0, 1) .$$
 (2.8)

Finally, there is a constant  $N \in (0, +\infty)$  such that

$$\int_0^t ds \, e^{-Bs} \, \mu(t-s) \leqslant N \qquad \text{for } t \in [0, +\infty) \ . \tag{2.9}$$

(P6) P is a bilinear map such that

$$\mathcal{P}: \mathbf{F} \times \mathbf{F} \to \mathbf{F}_{-}, \qquad (v, w) \mapsto \mathcal{P}(v, w);$$
 (2.10)

we assume continuity of  $\mathcal{P}$ , which is equivalent to the existence of a constant  $K \in [0, +\infty)$  such that, for all  $v, w \in \mathbf{F}$ ,

$$\|\mathcal{P}(v,w)\|_{-} \leqslant K\|v\|\|w\| \ . \tag{2.11}$$

The initial value problem; some results of uniqueness and existence. The initial value problem with initial datum  $u_0 \in \mathbf{F}$  is the following one:

Find 
$$u \in C([0,T), \mathbf{F})$$
 such that

$$u(t) = e^{tA}u_0 + \int_0^t ds \ e^{(t-s)A} \mathcal{P}(u(s), u(s)) \quad \text{for all } t \in [0, T) \ .$$
 (2.12)

Of course, we say that a solution u is maximal if it cannot be extended and global if its domain is  $[0, +\infty)$ . The following facts are well known.

- (i) With a bit more regularity ( $u_0 \in \mathbf{F}_+$  and  $\mathbf{F}_-$  reflexive), (2.12) is equivalent to the Cauchy problem  $du/dt = \mathcal{A}u + \mathcal{P}(u,u)$ ,  $u(0) = u_0$  for an unknown function  $u \in C([0,T),\mathbf{F}_+) \cap C^1([0,T),\mathbf{F}_-)$ . Independently of these stronger assumptions, a solution u of (2.12) is usually called a "mild solution" of the Cauchy problem.
- (ii) (2.12) has a unique maximal solution (and any other solution is a restriction of the maximal one);
- (iii) (2.12) has a global solution if the initial datum is small. Among the many references available on this point, for convenience we refer to our work [9], Proposition 5.12, yielding the following statement:

#### 2.1 Proposition. Suppose

$$4KN||u_0|| \leqslant 1$$
; (2.13)

then, the problem (2.12) has a global solution  $u:[0,+\infty)\to \mathbf{F}$ . Furthermore,

$$||u(t)|| \le \mathcal{X}(4KN||u_0||) e^{-Bt} ||u_0|| \quad for \ t \in [0, +\infty) ,$$
 (2.14)

where  $\mathcal{X} \in C([0,1],[1,2])$  is the increasing function defined by

$$\mathcal{X}(z) := \begin{cases} \frac{1 - \sqrt{1 - z}}{(z/2)} & \text{for } z \in (0, 1], \\ 1 & \text{for } z = 0. \end{cases}$$
 (2.15)

More roughly, due to the features of the function  $\mathcal{X}$ , one has

$$||u(t)|| \le 2 e^{-Bt} ||u_0|| \quad for \ t \in [0, +\infty)$$
 (2.16)

Let us mention that the global solution u in Proposition 2.1 is obtained as the limit of a Picard iteration:  $u(t) = \lim_{k \to +\infty} u_k(t)$  for each  $t \in [0, +\infty)$ , where  $(u_k)_{k=0,1,2,...}$  is the sequence of functions in  $C([0, +\infty), \mathbf{F})$  defined by  $u_0(t) := 0$  and  $u_{k+1}(t) := e^{tA}u_0 + \int_0^t ds \, e^{(t-s)A} \mathcal{P}(u_k(s), u_k(s))$ .

## 3 Sobolev spaces and the Navier-Stokes equations on a torus

In this section we consider any space dimension

$$d \geqslant 2. \tag{3.1}$$

We use r, s as indices running from 1 to d; elements a, b, ... of  $\mathbf{R}^d$  or  $\mathbf{C}^d$  are written with upper or lower indices, according to convenience:  $(a^r)$  or  $(a_r)$ ,  $(b^r)$  or  $(b_r)$ . For  $a, b \in \mathbf{C}^d$  (say, with upper indices), we put

$$a \cdot b := \sum_{r=1}^{d} a^r b^r ; \qquad |a| := \sqrt{\overline{a} \cdot a}$$
 (3.2)

where  $\overline{a} := (\overline{a^r})$  is the complex conjugate of a. Hereafter we refer to the d-dimensional torus

$$\mathbf{T}^{d} := \underbrace{\mathbf{T} \times ... \times \mathbf{T}}_{d \text{ times}} , \qquad \mathbf{T} := \mathbf{R}/(2\pi \mathbf{Z}) , \qquad (3.3)$$

whose elements are typically written  $x = (x^r)_{r=1,...d}$ .

Distributions on  $\mathbf{T}^d$ , Fourier series and Sobolev spaces. We introduce the space of periodic distributions  $D'(\mathbf{T}^d, \mathbf{C}) \equiv D'_{\mathbf{C}}$ , which is the dual of  $C^{\infty}(\mathbf{T}^d, \mathbf{C}) \equiv C^{\infty}_{\mathbf{C}}$  (equipping the latter with the topology of uniform convergence of all derivatives);  $\langle v, f \rangle$  denotes the action of a distribution  $v \in D'_{\mathbf{C}}$  on a test function  $f \in C^{\infty}_{\mathbf{C}}$ . We also consider the lattice  $\mathbf{Z}^d$  of elements  $k = (k_r)_{r=1,\dots,d}$  and the Fourier basis  $(e_k)_{k \in \mathbf{Z}^d}$ , where

$$e_k : \mathbf{T}^d \to \mathbf{C} , \qquad e_k(x) := \frac{1}{(2\pi)^{d/2}} e^{ik \bullet x}$$
 (3.4)

 $(k \cdot x = \sum_{r=1}^{d} k_r x^r)$  makes sense as an element of  $\mathbf{T}^d$ ). Each  $v \in D'_{\mathbf{C}}$  has a unique (weakly convergent) Fourier series expansion

$$v = \sum_{k \in \mathbf{Z}^d} v_k e_k , \qquad v_k := \langle v, e_{-k} \rangle \in \mathbf{C} .$$
 (3.5)

(As well known, the Fourier coefficients  $v_k$  of any  $v \in D'_{\mathbf{C}}$  grow polynomially in |k|; conversely, any family  $(v_k)_{k \in \mathbf{Z}^d}$  of complex numbers with such a polynomial growth is the family of the Fourier coefficients of some  $v \in D'_{\mathbf{C}}$ ). The mean of  $v \in D'_{\mathbf{C}}$  is

$$\langle v \rangle := \frac{1}{(2\pi)^d} \langle v, 1 \rangle = \frac{1}{(2\pi)^{d/2}} v_0 \tag{3.6}$$

(of course,  $\langle v, 1 \rangle = \int_{\mathbf{T}^d} dx \, v(x)$  if  $v \in L^1(\mathbf{T}^d)$ ); the space of zero mean distributions

$$D'_{G_0} := \{ v \in D'_{G_0} \mid \langle v \rangle = 0 \} . \tag{3.7}$$

The distributions v in this subspace are characterized by the equivalent condition  $v_0 = 0$ ; so, their relevant Fourier coefficients are labeled by the set

$$\mathbf{Z}_0^d := \mathbf{Z}^d \setminus \{0\} \ . \tag{3.8}$$

The complex conjugate of a distribution  $v \in D'_{\mathbf{C}}$  is the unique distribution  $\overline{v}$  such that  $\overline{\langle v, f \rangle} = \langle \overline{v}, \overline{f} \rangle$  for each  $f \in C_{\mathbf{C}}^{\infty}$ ; one has  $\overline{v} = \sum_{k \in \mathbf{Z}^d} \overline{v_k} \, e_{-k}$ .

The distributional derivatives  $\partial/\partial x^s \equiv \partial_s \ (s=1,...,d)$  and the Laplacian  $\Delta:=$  $\sum_{s=1}^{d} \partial_{ss}$  obviously send  $D'_{\mathbf{C}}$  in  $D'_{\mathbf{C}_0}$ , and are such that, for any  $v \in D_{\mathbf{C}}$ ,  $\partial_s v = i \sum_{k \in \mathbf{Z}_0^d} k_s v_k e_k$ ,  $\Delta v = -\sum_{k \in \mathbf{Z}_0^d} |k|^2 v_k e_k$ . For any  $n \in \mathbf{R}$ , we further define

$$\sqrt{-\Delta}^n : D'_{\mathbf{C}} \to D'_{\mathbf{C}_0} , \qquad v \mapsto \sqrt{-\Delta}^n v := \sum_{k \in \mathbf{Z}_0^d} |k|^n v_k e_k . \tag{3.9}$$

In the sequel, we are interested in the space of real distributions  $D'(\mathbf{T}^d, \mathbf{R}) \equiv D'$ , defined as follows:

$$D' := \{ v \in D'_{\mathbf{C}} \mid \overline{v} = v \} = \{ v \in D'_{\mathbf{C}} \mid \overline{v_k} = v_{-k} \text{ for all } k \in \mathbf{Z}^d \} ;$$
 (3.10)

of course,  $v \in D'$  implies  $\langle v \rangle \in \mathbf{R}$ . We also set

$$D_0' := \{ v \in D' \mid \langle v \rangle = 0 \} ; \tag{3.11}$$

all the differential operators mentioned before send D' into  $D'_0$ . Hereafter we consider the real Hilbert space  $L^2(\mathbf{T}^d, \mathbf{R}, dx) \equiv L^2$ , with the inner product  $\langle v|w\rangle_{L^2} := \int_{\mathbf{T}^d} v(x)w(x)dx = \sum_{k\in\mathbf{Z}^d} \overline{v_k}w_k$  and the induced norm

$$||v||_{L^2} = \sqrt{\int_{\mathbf{T}^d} v^2(x) dx} = \sqrt{\sum_{k \in \mathbf{Z}^d} |v_k|^2}$$
 (3.12)

To go on, we introduce the zero mean Sobolev spaces  $H_0^n(\mathbf{T}^d,\mathbf{R}) \equiv H_0^n$ . For each  $n \in \mathbf{R}$ 

$$H_0^n := \{ v \in D_0' \mid \sqrt{-\Delta}^n v \in L^2 \} = \{ v \in D_0' \mid \sum_{k \in \mathbf{Z}_0^d} |k|^{2n} |v_k|^2 < +\infty \} ; \qquad (3.13)$$

this is a real Hilbert space with inner product  $\langle v|w\rangle_n:=\langle\sqrt{-\Delta}^nv|\sqrt{-\Delta}^nw\rangle_{L^2}$  $=\sum_{k\in\mathbf{Z}_0^d}|k|^{2n}\,\overline{v_k}w_k$  and the induced norm

$$||v||_n = ||\sqrt{-\Delta}^n v||_{L^2} = \sqrt{\sum_{k \in \mathbf{Z}_0^d} |k|^{2n} |v_k|^2}$$
 (3.14)

Let us consider, in particular, the case n is a nonnegative integer. For all  $v \in D'$  and  $k \in \mathbf{Z}^d$ , one has  $|k|^{2n}|v_k|^2 = (\sum_{s=1}^d k_s^2)^n|v_k|^2 = \sum_{s_1,...,s_n=1}^d k_{s_1}^2...k_{s_n}^2|v_k|^2 = \sum_{s_1,...,s_n=1}^d |(\partial_{s_1...s_n}v)_k|^2$ ; so,

$$H_0^n := \{ v \in D_0' \mid \partial_{s_1...s_n} v \in L^2 \ \forall s_1, ..., s_n \in \{1, ..., d\} \} ;$$
 (3.15)

$$||v||_n = \sqrt{\sum_{s_1, \dots, s_n = 1}^d ||\partial_{s_1 \dots s_n} v||_{L^2}^2} \qquad \forall v \in H_0^n . \tag{3.16}$$

All these details about Sobolev spaces may seem pleonastic, but in fact they are useful to ensure that some norms we use later coincide exactly with the norms of [10].

For (integer or noninteger)  $n' \leqslant n$ , one has  $H_0^n \hookrightarrow H_0^{n'}$  and  $\| \|_{n'} \leqslant \| \|_n$  on  $H_0^n$ . In particular,  $H_0^0$  is the subspace of  $L^2$  made of zero mean elements. For any real n,  $\Delta H_0^n = H_0^{n-2}$  and  $\Delta$  is continuous between these spaces. Obviously enough, we could define as well the complex Hilbert spaces  $L_{\mathbb{C}}^2$  and  $H_{\mathbb{C}_0}^n$ ; however, these are never needed in the sequel. Other facts about Sobolev spaces (in particular, the duality between  $H_0^{-n}$  and  $H_0^n$ ) are mentioned when necessary in the sequel.

Spaces of vector valued functions on  $\mathbf{T}^d$ . If  $V(\mathbf{T}^d, \mathbf{R}) \equiv V$  is any vector space of real functions or distributions on  $\mathbf{T}^d$ , we write

$$\mathbb{V}(\mathbf{T}^d) \equiv \mathbb{V} := \{ v = (v^1, ..., v^d) \mid v^r \in V \text{ for all } r \}.$$
 (3.17)

In this way we can define, e.g., the spaces  $\mathbb{D}'(\mathbf{T}^d) \equiv \mathbb{D}'$ ,  $\mathbb{L}^2(\mathbf{T}^d) \equiv \mathbb{L}^2$ ,  $\mathbb{H}^n_0(\mathbf{T}^d) \equiv \mathbb{H}^n_0$ . Any  $v = (v^r) \in \mathbb{D}'$  is referred to as a (distributional) vector field on  $\mathbf{T}^d$ . We note that v has a unique Fourier series expansion (3.5) with coefficients

$$v_k = (v_k^r)_{r=1,\dots,d} \in \mathbf{C}^d , \qquad v_k^r := \langle v^r, e_{-k} \rangle ;$$
 (3.18)

as in the scalar case, the reality of v ensures  $\overline{v_k} = v_{-k}$ . We define componentwisely the mean  $\langle v \rangle \in \mathbf{R}^d$  of any  $v \in \mathbb{D}'$  (see Eq. (3.6));  $\mathbb{D}'_0$  is the space of zero mean vector fields. We similarly define componentwisely the operators  $\partial_s, \Delta, \sqrt{-\Delta}^n$ :  $\mathbb{D}' \to \mathbb{D}'_0$ .  $\mathbb{L}^2$  is a real Hilbert space; its inner product is as in the line before (3.12), with v(x)w(x) and  $\overline{v_k}w_k$  replaced by  $v(x) \cdot w(x) = \sum_{r=1}^d v^r(x) w^r(x)$  and  $\overline{v_k} \cdot w_k = \sum_{r=1}^d \overline{v_k^r} w_k^r$ . For any real n, the n-th Sobolev space of zero mean vector fields  $\mathbb{H}^n_0(\mathbf{T}^d) \equiv \mathbb{H}^n_0$  is made of all d-uples v with components  $v^r \in H^n_0$ ; an equivalent definition can be given via Eq.(3.13), replacing therein  $L^2$  with  $\mathbb{L}^2$ .  $\mathbb{H}^n_0$  is a real Hilbert space with the inner product  $\langle v|w\rangle_n := \langle \sqrt{-\Delta}^n v | \sqrt{-\Delta}^n w\rangle_{L^2} = \sum_{k \in \mathbf{Z}^n_0} |k|^{2n} \overline{v_k} \cdot w_k$ ; the induced norm  $\|\cdot\|_n$  is given, verbatim, by Eq. (3.14). If  $w \in \mathbb{H}^n_0$  has components  $w^r$ , we have obviously

$$||w||_n = \sqrt{\sum_{r=1}^d ||w^r||_n^2} ; (3.19)$$

for n a nonnegative integer, the vector analogue of Eq. (3.16) is

$$||w||_n = \sqrt{\sum_{r,s_1,\dots,s_n=1}^d ||\partial_{s_1\dots s_n} w^r||_{L^2}^2} .$$
 (3.20)

**Divergence free vector fields.** Let us consider the divergence operator div :  $\mathbb{D}' \to D'_0$ ,  $v \mapsto \operatorname{div} v := \sum_{r=1}^d \partial_r v^r$ ; of course,  $\operatorname{div} v = i \sum_{k \in \mathbb{Z}_0^d} (k \cdot v_k) e_k$ . The space of divergence free (or solenoidal) vector fields is

$$\mathbb{D}_{\Sigma}' := \{ v \in \mathbb{D}' \mid \operatorname{div} v = 0 \} = \{ v \in \mathbb{D}' \mid k \cdot v_k = 0 \ \forall k \in \mathbf{Z}^d \} ; \tag{3.21}$$

in the sequel, we consider as well the spaces

$$\mathbb{D}'_{\Sigma_0} := \mathbb{D}'_{\Sigma} \cap \mathbb{D}'_0 , \qquad \mathbb{H}^n_{\Sigma_0} := \mathbb{D}'_{\Sigma} \cap \mathbb{H}^n_0 \quad (n \in \mathbf{R})$$
 (3.22)

(as usually, reference to  $\mathbf{T}^d$  is omitted for brevity: for example,  $\mathbb{H}^n_{\Sigma^0}$  stands for  $\mathbb{H}^n_{\Sigma^0}(\mathbf{T}^d)$ ).  $\mathbb{H}^n_{\Sigma^0}$  is a closed subspace of the Hilbert space  $\mathbb{H}^n_0$ , equipped with the restriction of the inner product  $\langle \ | \ \rangle_n$ . We note the relations  $\Delta(\mathbb{D}'_{\Sigma}) = \mathbb{D}'_{\Sigma^0}$  and  $\Delta(\mathbb{H}^{n+2}_{\Sigma}) = \mathbb{H}^n_{\Sigma^0}$ , following immediately from the Fourier representations. The *Leray projection* is the map

$$\mathfrak{L}: \mathbb{D}' \to \mathbb{D}'_{\Sigma}, \qquad v \mapsto \mathfrak{L}v := \sum_{k \in \mathbf{Z}^d} (\mathfrak{L}_k v_k) e_k,$$
 (3.23)

where, for each k,  $\mathcal{L}_k$  is the orthogonal projection of  $\mathbf{C}^d$  onto the orthogonal complement of k; more explicitly, if  $c \in \mathbf{C}^d$ ,

$$\mathfrak{L}_0 c = c$$
,  $\mathfrak{L}_k c = c - \frac{k \cdot c}{|k|^2} k$  for  $k \in \mathbf{Z}_0^d$ . (3.24)

From the Fourier representations of  $\mathfrak{L}$ ,  $\langle \ \rangle$ ,  $\partial_s$ , etc., one easily infers the following statements:

$$\langle \mathfrak{L}v \rangle = \langle v \rangle, \quad \mathfrak{L}(\partial_s v) = \partial_s(\mathfrak{L}v), \quad \mathfrak{L}(\Delta v) = \Delta(\mathfrak{L}v) \quad \forall v \in \mathbb{D}';$$
 (3.25)

$$\mathfrak{L}\mathbb{D}_0' = \mathbb{D}_{\Sigma_0}', \quad \mathfrak{L}\mathbb{H}_0^n = \mathbb{H}_{\Sigma_0}^n; \quad \|\mathfrak{L}v\|_n \leqslant \|v\|_n \ \forall n \in \mathbf{R}, v \in \mathbb{H}_0^n \ . \tag{3.26}$$

The spaces  $\mathbb{H}_0^1$ ,  $\mathbb{H}_{\Sigma_0}^1$ . In the sequel, we are often interested in specializing the previous considerations to the case n=1. This carries us to the space

$$\mathbb{H}_{0}^{1} := \{ v \in \mathbb{D}_{0}' \mid \sqrt{-\Delta} v \in L^{2} \} = \{ v \in \mathbb{D}_{0}' \mid \partial_{s} v^{r} \in L^{2} \ \forall r, s \in \{1, ..., d\} \}$$

$$= \{ v \in \mathbb{D}_{0}' \mid \sum_{k \in \mathbf{Z}_{0}^{d}} |k|^{2} |v_{k}|^{2} < +\infty \} , \qquad (3.27)$$

with the norm

$$||v||_1 = ||\sqrt{-\Delta} v||_{L^2} = \sqrt{\sum_{r,s=1}^d ||\partial_s v^r||_{L^2}^2} = \sqrt{\sum_{k \in \mathbf{Z}_0^d} |k|^2 |v_k|^2} . \tag{3.28}$$

We are as well interested in the divergence free part  $\mathbb{H}^1_{\Sigma_0} = \mathbb{D}'_{\Sigma} \cap \mathbb{H}^1_0$ . We note that

$$||v||_1 = ||\operatorname{curl} v||_{L^2} \quad \text{if } d = 3 \text{ and } v \in \mathbb{H}^1_{\Sigma_0},$$
 (3.29)

where the right hand side contains the curl of vector fields (i.e.,  $(\operatorname{curl} v)^1 := \partial_2 v^3$  $\partial_3 v^2$ , etc., intending derivatives in the distributional sense). To check Eq. (3.29), one notes that  $(\operatorname{curl} v)_k = ik \wedge v_k$  where  $\wedge$  is the (complexified) vector product (so that  $(k \wedge a)^1 := k_2 a^3 - k_3 a^2$ , etc., for all  $a \in \mathbb{C}^3$ ); if v is divergence free, the orthogonality property  $k \cdot v_k = 0$  implies  $|k \wedge v_k| = |k||v_k|$ , whence  $\|\operatorname{curl} v\|_{L^2}^2 = \sum_{k \in \mathbb{Z}_0^3} |k|^2 |v_k|^2 =$  $||v||_1^2$ .

(Before going on, we point out that the norm  $||v||_1$  on  $\mathbb{H}^1_{\Sigma_0}$  (in dimension d=3) is denoted in [10] with ||Dv||, the symbol || || standing for the  $L^2$  norm: see page 40 of the cited reference. The equality  $||v||_1 = ||Dv||$  is essential for subsequent comparison between some estimates of ours and [10]).

### The exponential of the Laplacian. Let us put

$$e^{t\Delta}v := \sum_{k \in \mathbf{Z}^d} e^{-t|k|^2} v_k e_k \qquad \forall t \in [0, +\infty), v \in \mathbb{D}' ;$$
 (3.30)

the above series converges in the weak topology of  $\mathbb{D}'$ . From the Fourier representations, it is clear that

$$e^{t\Delta}\mathbb{D}_0' \subset \mathbb{D}_0'$$
,  $e^{t\Delta}\mathbb{D}_{\Sigma_0}' \subset \mathbb{D}_{\Sigma_0}' \quad \forall t \in [0, +\infty).$  (3.31)

In the sequel, we consider any  $n \in \mathbf{R}$ . As well known, the map  $(t, v) \mapsto e^{t\Delta}v$  sends continuously  $[0,+\infty)\times\mathbb{H}_0^n$  into  $\mathbb{H}_0^n$ ; this map is a strongly continuous semigroup on the Hilbert space  $\mathbb{H}_0^n$ , with generator  $\Delta \upharpoonright \mathbb{H}_0^{n+2} : \mathbb{H}_0^{n+2} \to \mathbb{H}_0^n$ . Furthermore

$$||e^{t\Delta}v||_n \leqslant e^{-t}||v||_n \qquad \forall t \in [0, +\infty), v \in \mathbb{H}_0^n \tag{3.32}$$

(as follows immediately from the Fourier representations (3.30) of  $e^{t\Delta}$ , combined with the obvious inequality  $e^{-t|k|^2} \leq e^{-t}$  for all  $k \in \mathbf{Z}_0^d$ ). Now, let us consider two Sobolev spaces  $\mathbb{H}_0^n$ ,  $\mathbb{H}_0^{n-\nu}$ , with  $\nu \in (0, +\infty)$ ; then

$$e^{t\Delta}v \in \mathbb{H}_0^n$$
,  $\|e^{t\Delta}v\|_n \leqslant \hat{\mu}_{\nu}(t)e^{-t}\|v\|_{n-\nu}$   $\forall t \in (0, +\infty), v \in \mathbb{H}_0^{n-\nu}$ , (3.33)

$$\hat{\mu}_{\nu}(t) := \begin{cases} \left(\frac{\nu}{2et}\right)^{\frac{\nu}{2}} e^{t} & \text{for } 0 < t \leq \frac{\nu}{2} ,\\ 1 & \text{for } t > \frac{\nu}{2} ; \end{cases}$$

$$(3.34)$$

(see Appendix A). The map  $(t,v) \mapsto e^{t\Delta}v$  sends continuously  $(0,+\infty) \times \mathbb{H}_0^{n-\nu}$  into  $\mathbb{H}_0^n$ .

All the previous statements about  $e^{t\Delta}$  and the Sobolev spaces, including the relations (3.32)-(3.34), hold as well if we replace systematically  $\mathbb{H}^n_0$ ,  $\mathbb{H}^{n+2}_0$ ,  $\mathbb{H}^{n-\nu}_0$  with  $\mathbb{H}^n_{\Sigma 0}$ ,  $\mathbb{H}^{n+2}_{\Sigma 0}$ ,  $\mathbb{H}^{n-\nu}_{\Sigma 0}$ ; this is basically due to the relations  $\Delta \mathbb{D}'_{\Sigma 0} = \mathbb{D}'_{\Sigma 0}$  and  $e^{t\Delta} \mathbb{D}'_{\Sigma 0} \subset \mathbb{D}'_{\Sigma 0}$ .

**NS equations.** A setting for the incompressible NS equations on  $\mathbf{T}^d$  consists of three Banach spaces  $\mathbf{F}_+$ ,  $\mathbf{F}$ ,  $\mathbf{F}_-$  of divergence free vector fields on  $\mathbf{T}^d$ , such that all conditions (P1-P6) of Section 2 are fulfilled by these spaces, by the operator

$$\mathcal{A} := \Delta \upharpoonright \mathbf{F}_{+} : \mathbf{F}_{+} \mapsto \mathbf{F}_{-} \tag{3.35}$$

and by the bilinear map

$$\mathcal{P}: \mathbf{F} \times \mathbf{F} \to \mathbf{F}_{-}, \qquad (v, w) \mapsto \mathcal{P}(v, w) := -\mathfrak{L}(v \cdot \partial w). \tag{3.36}$$

In the above,  $v \cdot \partial w$  is the vector field on  $\mathbf{T}^d$  with components  $(v \cdot \partial w)^r := \sum_{s=1}^d v^s \partial_s w^r$ ; the function spaces are chosen so that, for  $v, w \in \mathbf{F}$ , the pointwise products  $v^s \partial_s w^r$  are well defined and  $\mathcal{P}(v, w)$  belongs to  $\mathbf{F}_-$ .

The initial value problem (2.12) corresponding to the above maps  $\mathcal{A}, \mathcal{P}$  and to any datum  $u_0 \in \mathbf{F}$  takes the form

Find 
$$u \in C([0,T), \mathbf{F})$$
 such that

$$u(t) = e^{t\Delta}u_0 - \int_0^t ds \ e^{(t-s)\Delta} \mathfrak{L}(u(s) \bullet \partial u(s)) \quad \text{for all } t \in [0, T) \ , \tag{3.37}$$

and is related, in the sense already indicated, to the Cauchy problem  $du/dt = \Delta u - \mathfrak{L}(u \cdot \partial u)$ ,  $u(0) = u_0$ . One recognizes the NS evolution equation for an incompressible fluid (in units in which the density and the viscosity are 1, and assuming no external forces).

In [9] we considered for arbitrary d the setting  $\mathbf{F} := \mathbb{H}^n_{\Sigma 0}$ ,  $\mathbf{F}_{\pm} := \mathbb{H}^{n\pm 1}_{\Sigma 0}$  ( $n \in (d/2, +\infty)$ ), putting the emphasis on all the quantitative aspects and, in particular, on the accurate estimation of the constants N, K in Section 2; this allowed to get fully quantitative estimates on the time of existence for the solution of (2.12), and gave a framework to evaluate a posteriori the distance between any approximate solution and the exact solution.

As anticipated in the Introduction, here we will take a similar attitude for the setting

$$\mathbf{F}_{+} := \mathbb{H}_{\Sigma_{0}}^{2-\omega} , \quad \mathbf{F} := \mathbb{H}_{\Sigma_{0}}^{1} , \quad \mathbf{F}_{-} := \mathbb{H}_{\Sigma_{0}}^{-\omega} ,$$
 (3.38)

mainly for d=3 and  $1/2 < \omega < 1$ . Again, our evaluation of N, K will be relevant in connection with the time of existence for (3.37), with the estimates about approximate solutions, and so on.

## 4 An $H^1$ setting for the semigroup of the Laplacian

For the moment we assume

$$d \in \{2, 3, \dots\}$$
,  $\omega \in (0, 1)$  (4.1)

and take  $\mathbf{F}_-$ ,  $\mathbf{F}$ ,  $\mathbf{F}_+$  as in (3.38),  $\mathcal{A}$  as in (3.35); the inclusion (P1)  $\mathbf{F}_- \hookrightarrow \mathbf{F} \hookrightarrow \mathbf{F}_+$  is evident.

**4.1** Proposition. (i) A generates a strongly continuous semigroup on  $\mathbb{H}_{\Sigma_0}^{-\omega} = \mathbf{F}_{-}$ ; this is the restriction to  $[0, +\infty) \times \mathbb{H}_{\Sigma_0}^{-\omega}$  of the map  $(t, v) \mapsto e^{t\Delta}v$  (see Eq. (3.30)). (ii) The map  $(t, v) \mapsto e^{t\Delta}v$  gives as well a strongly continuous semigroup on  $\mathbb{H}_{\Sigma_0}^1 = \mathbf{F}$ ; one has

$$||e^{t\Delta}v||_1 \leqslant e^{-t}||v||_1 \qquad \forall t \in [0, +\infty), v \in \mathbb{H}^1_{\Sigma_0}$$
 (4.2)

(iii) The map  $(t, v) \mapsto e^{t\Delta}v$  sends continuously  $(0, +\infty) \times \mathbb{H}_{\Sigma_0}^{-\omega} = (0, +\infty) \times \mathbf{F}_-$  into  $\mathbb{H}_{\Sigma_0}^1 = \mathbf{F}$ ; one has

$$||e^{t\Delta}v||_1 \leqslant \mu_{\omega}(t)e^{-t}||v||_{-\omega} \qquad \forall t \in (0, +\infty), v \in \mathbb{H}_{\Sigma_0}^{-\omega}, \qquad (4.3)$$

$$\mu_{\omega}(t) := \begin{cases} \left(\frac{1+\omega}{2et}\right)^{\frac{1+\omega}{2}} e^{t} & for \quad 0 < t \leqslant \frac{1+\omega}{2} ,\\ 1 & for \quad t > \frac{1+\omega}{2} . \end{cases}$$

$$(4.4)$$

(Note that  $\mu_{\omega}(t) = O(1/t^{(1+\omega)/2})$  for  $t \to 0^+$ ; this power of 1/t is integrable due to the assumption  $0 < \omega < 1$ ). With  $\mu_{\omega}$  as above, one has

$$\sup_{t \in [0, +\infty)} \int_0^t ds \, e^{-s} \, \mu_\omega(t - s) < +\infty \ . \tag{4.5}$$

(iv) Statements (i-iii) indicate that the set  $(\mathbf{F}_+, \mathbf{F}, \mathbf{F}_-, \mathcal{A})$  fulfills all conditions (P2-P5) of Section 2. The function  $\mu$  and the constants  $B, N \in (0, +\infty)$  mentioned therein can be taken as follows:

$$\mu = \mu_{\omega}$$
 as in (4.4),  $B = 1$ ,  $N \equiv N_{\omega} := any majorant of the sup (4.5). (4.6)$ 

**Proof.** (i-iii) Essentially, one specializes to the present case all statements of Section 3 (and Appendix A) about  $(e^{t\Delta})$  and Sobolev spaces of arbitrary order. In particular, the relations (4.2)-(4.4) follow from the relations (3.32)-(3.34) for the spaces  $\mathbb{H}^n_{\Sigma^0}$  and  $\mathbb{H}^{n-\nu}_{\Sigma^0}$ , with n=1 and  $\nu=1+\omega$ . With this choice of  $\nu$ , the function  $\hat{\mu}_{\nu}$  of Eq. (3.34) becomes the  $\mu_{\omega}$  of Eq. (4.4); the finiteness of the sup (4.5) is easily inferred from the explicit expression of  $\mu_{\omega}$ , by an argument very similar to the one employed in [7] to prove Proposition 7.2, item (iv).

(iv) Obvious. 
$$\Box$$

## 5 An $H^1$ setting for the NS quadratic nonlinearity

Throughout this section we assume

$$d \in \{2, 3, ...\}$$
,  $\omega \in (\frac{d}{2} - 1, +\infty)$ . (5.1)

Our aim is to show that the NS bilinear map  $(v, w) \mapsto \mathcal{P}(v, w) := -\mathfrak{L}(v \cdot \partial w)$  is well defined from  $\mathbf{F} \times \mathbf{F} = \mathbb{H}^1_{\Sigma_0} \times \mathbb{H}^1_{\Sigma_0}$  to  $\mathbf{F}_- = \mathbb{H}^{-\omega}_{\Sigma_0}$ , and to determine a constant  $K_{\omega}$  such that  $\|\mathcal{P}(v, w)\|_{-\omega} \leqslant K_{\omega} \|v\|_1 \|w\|_1$ .

We will reach this result through a number of intermediate steps, most of them relying on the function

$$\mathcal{K}_{\omega}: \mathbf{Z}_{0}^{d} \to (0, +\infty) , \qquad k \mapsto \mathcal{K}_{\omega}(k) := \sum_{h \in \mathbf{Z}_{0k}^{d}} \frac{1}{|h|^{2\omega} |k - h|^{2}} ; \qquad (5.2)$$

in the above one should intend

$$\mathbf{Z}_{0k}^d := \mathbf{Z}^d \setminus \{0, k\} , \qquad (5.3)$$

a notation to be employed systematically in the sequel.

In Appendix C we show the following.

- (i) The sum defining  $\mathcal{K}_{\omega}(k)$  (in principle, existing in  $(0, +\infty]$ ) is in fact finite, for each  $k \in \mathbf{Z}_0^d$ .
- (ii) One has  $\sup_{k \in \mathbf{Z}_0^d} \mathcal{K}_{\omega}(k) < +\infty$ .
- (iii) Explicit lower and upper bounds can be given for  $\mathcal{K}_{\omega}$ , to be used for the practical evaluation of this function and of its sup.

The function  $\mathcal{K}_{\omega}$  first appears in the forthcoming Proposition, dealing with pointwise multiplication in the Sobolev spaces (of *scalar* functions)  $H_0^{\omega}$ ,  $H_0^1$ ; this Proposition will be subsequently employed to estimate the NS bilinear map.

### **5.1** Proposition. With d and $\omega$ as in (5.1), the following holds.

(i) Let  $z \in H_0^{\omega}$ ,  $v \in H_0^1$ , so that zv (the pointwise product of two  $L^2$  functions) is well defined (and in  $L^1$ ). This product has the additional features

$$zv \in L^2$$
,  $||zv - \langle zv \rangle||_{L^2} \leqslant K_\omega ||z||_\omega ||v||_1$  (5.4)

for a suitable constant  $K_{\omega} \in (0, +\infty)$ , independent of z, v (recall that  $\langle \rangle$  is the mean).

(ii) The above inequality is fulfilled for all z, v by any constant  $K_{\omega}$  such that

$$\frac{1}{(2\pi)^{d/2}} \sqrt{\sup_{k \in \mathbf{Z}_0^d} \mathcal{K}_{\omega}(k)} \leqslant K_{\omega}, \tag{5.5}$$

with  $\mathcal{K}_{\omega}$  as in Eq. (5.2).

**Proof.** Let us consider the Fourier coefficients  $(zv)_k$  of zv, for  $k \in \mathbb{Z}^d$ ; up to a factor  $1/(2\pi)^{d/2}$ , these are obtained taking the convolution of the Fourier coefficients of z, v:

$$(zv)_k = \frac{1}{(2\pi)^{d/2}} \sum_{h \in \mathbf{Z}^d} z_h v_{k-h} = \frac{1}{(2\pi)^{d/2}} \sum_{h \in \mathbf{Z}_{c}^d} z_h v_{k-h}$$
 (5.6)

(the sum can be confined to  $\mathbf{Z}_{0k}^d$ , due to the vanishing of  $z_0, v_0$ ). From here we get

$$|(zv)_k| \leqslant \frac{1}{(2\pi)^{d/2}} \sum_{h \in \mathbf{Z}_{0k}^d} |z_h| |v_{k-h}|$$
 (5.7)

$$= \frac{1}{(2\pi)^{d/2}} \sum_{h \in \mathbf{Z}_{0h}^d} \frac{1}{|h|^{\omega}|k-h|} |h|^{\omega}|z_h||k-h||v_{k-h}|.$$

Now, Hölder's inequality  $|\sum_h a_h b_h|^2 \leqslant \left(\sum_h |a_h|^2\right) \left(\sum_h |b_h|^2\right)$  gives

$$|(zv)_k|^2 \leqslant \frac{1}{(2\pi)^d} \mathcal{K}_{\omega}(k) \mathcal{P}(k) , \qquad (5.8)$$

$$\mathcal{K}_{\omega}(k) := \sum_{h \in \mathbf{Z}_{0k}^d} \frac{1}{|h|^{2\omega}|k-h|^2} \text{ as in } (5.2), \quad \mathcal{P}(k) := \sum_{h \in \mathbf{Z}_{0k}^d} |h|^{2\omega}|z_h|^2|k-h|^2|v_{k-h}|^2.$$

Eq. (5.8) implies

$$\sum_{k \in \mathbf{Z}_0^d} |(zv)_k|^2 \leqslant \frac{1}{(2\pi)^d} \sum_{k \in \mathbf{Z}_0^d} \mathcal{K}_{\omega}(k) \mathcal{P}(k)$$

$$\leqslant \frac{1}{(2\pi)^d} \Big( \sup_{k \in \mathbf{Z}_0^d} \mathcal{K}_{\omega}(k) \Big) \sum_{k \in \mathbf{Z}_0^d} \mathcal{P}(k) \leqslant K_{\omega}^2 \sum_{k \in \mathbf{Z}_0^d} \mathcal{P}(k)$$
(5.9)

where  $K_{\omega}$  is any constant such that  $(2\pi)^{-d/2}\sqrt{\sup_{k\in\mathbb{Z}_0^d}\mathcal{K}_{\omega}(k)} \leqslant K_{\omega}$ , as in (5.5). To continue, let us observe that the definition of  $\mathcal{P}$  implies

$$\sum_{k \in \mathbf{Z}_0^d} \mathcal{P}(k) = \sum_{(k,h) \in \mathbf{Z}_0^d \times \mathbf{Z}_0^d, k \neq h} |h|^{2\omega} |z_h|^2 |k-h|^2 |v_{k-h}|^2$$

whence, with a change of variable  $\ell = k - h$ ,

$$\sum_{k \in \mathbf{Z}_0^d} \mathcal{P}(k) = \sum_{(h,\ell) \in \mathbf{Z}_0^d \times \mathbf{Z}_0^d} |h|^{2\omega} |z_h|^2 |\ell|^2 |v_\ell|^2 = ||z||_{\omega}^2 ||v||_1^2.$$
 (5.10)

From Eqs.(5.9) (5.10), we get

$$\sum_{k \in \mathbf{Z}_0^d} |(zv)_k|^2 \leqslant K_\omega^2 ||z||_\omega^2 ||v||_1^2 . \tag{5.11}$$

The finiteness of  $\sum_{k \in \mathbf{Z}_0^d} |(zv)_k|^2$ , and thus of  $\sum_{k \in \mathbf{Z}^d} |(zv)_k|^2$ , ensures  $zv \in L^2$ . Noting that  $zv - \langle zv \rangle = \sum_{k \in \mathbf{Z}_0^d} (zv)_k e_k$  we can reexpress (5.11) as

$$||zv - \langle zv \rangle||_{L^2}^2 \leqslant K_{\omega}^2 ||z||_{\omega}^2 ||v||_1^2 , \qquad (5.12)$$

whence the thesis.  $\Box$ 

**5.2 Proposition.** Let d,  $\omega$  be as in (5.1), and denote with  $K_{\omega}$  any constant fulfilling Eq. (5.5). Furthermore, let

$$v \in \mathbb{H}^1_{\Sigma_0}$$
,  $w \in \mathbb{H}^1_0$ ; (5.13)

then (i)(ii) hold.

(i) Consider the product  $v \cdot \partial w$  (whose components  $(v \cdot \partial w)^r := \sum_{s=1}^d v^s \partial_s w^r$  are well defined and belong to  $L^1$ , being sum of products of the  $L^2$  functions  $v^s$ ,  $\partial_s w^r$ ). One has

$$v \cdot \partial w \in \mathbb{H}_0^{-\omega}$$
,  $\|v \cdot \partial w\|_{-\omega} \leqslant K_{\omega} \|v\|_1 \|w\|_1$ . (5.14)

(ii) Furthermore, one has

$$\mathfrak{L}(v \bullet \partial w) \in \mathbb{H}_{\Sigma_0}^{-\omega} , \qquad \|\mathfrak{L}(v \bullet \partial w)\|_{-\omega} \leqslant K_{\omega} \|v\|_1 \|w\|_1 . \tag{5.15}$$

(Due to the above results, condition (P6) of Section 2 is fulfilled with  $\mathbf{F} := \mathbb{H}^1_{\Sigma_0}$ ,  $\mathbf{F}_- := \mathbb{H}^{-\omega}_{\Sigma_0}$ ,  $\mathfrak{P} : \mathbf{F} \times \mathbf{F} \to \mathbf{F}_-$ ,  $\mathfrak{P}(v,w) := -\mathfrak{L}(v \bullet \partial w)$  and  $K := K_\omega$ ).

**Proof.** Let us observe that (ii) follows immediately from (i) using (3.26) with  $n \to -\omega$  and  $v \to v \bullet \partial w$ . Hereafter, we give the proof of (i); after putting for brevity

$$p := v \cdot \partial w , \qquad (5.16)$$

we proceed in several steps.

Step 1. One has

$$p^r \in L_0^1 := L^1 \cap D_0' \tag{5.17}$$

for all  $r \in \{1, ..., d\}$  (i.e.,  $p \in \mathbb{L}_0^1 := \mathbb{L}^1 \cap \mathbb{D}_0'$ ). We have just noticed, as a comment in the statement of this Proposition, that  $p^r \in L^1$ ; furthermore,  $p^r$  has zero mean since

$$\int_{\mathbf{T}^d} p^r dx = \sum_{s=1}^d \int_{\mathbf{T}^d} v^s \partial_s w^r = -\sum_{s=1}^d \int_{\mathbf{T}^d} (\partial_s v^s) w^r = 0$$
 (5.18)

(here, we have used an integration by parts and the assumption divv = 0). Step 2. For all  $r \in \{1, ..., d\}$  one has

$$p^r \in H_0^{-\omega}$$
,  $||p^r||_{-\omega} \leqslant K_\omega ||v||_1 \sqrt{\sum_{s=1}^d ||\partial_s w^r||_{L^2}^2}$ . (5.19)

We know that  $p^r \in L_0^1$ ; by the familiar duality between  $H_0^{\omega}$  and  $H_0^{-\omega}$ ,

$$p^r \in H_0^{-\omega} \qquad \Longleftrightarrow \qquad zp^r \in L^1 \ \forall z \in H_0^{\omega}, \ \sup_{z \in H_0^{\omega}, z \neq 0} \frac{\left| \int_{\mathbf{T}^d} z p^r dx \right|}{\|z\|_{\omega}} < +\infty \ ; \quad (5.20)$$

if the above conditions are fulfilled, we further have

$$||p^r||_{-\omega} = \sup_{z \in H_0^{\omega}, z \neq 0} \frac{|\int_{\mathbf{T}^d} z p^r dx|}{||z||_{\omega}}$$
 (5.21)

Keeping in mind the above statements, we consider any  $z \in H_0^{\omega}$  and the function

$$zp^r = \sum_{s=1}^d z \, v^s \partial_s w^r \ . \tag{5.22}$$

For each s we have the following:

- (i)  $\partial_s w^r \in L^2$ , due to the assumption  $w \in \mathbb{H}_0^1$ ; (ii)  $zv^s \in L^2$ : this follows using Proposition 5.1, which also gives

$$||zv^s - \langle zv^s \rangle||_{L^2} \leqslant K_\omega ||z||_\omega ||v^s||_1$$
 (5.23)

From  $zv^s, \partial_s w^r \in L^2$  for all s, it follows

$$zp^r \in L^1 (5.24)$$

now we estimate the integral of  $zp^r$ , starting from the following remark:

$$\int_{\mathbf{T}^d} z p^r dx = \sum_{s=1}^d \int_{\mathbf{T}^d} (z v^s) \partial_s w^r$$
 (5.25)

$$= \sum_{s=1}^{d} \int_{\mathbf{T}^{d}} (zv^{s} - \langle zv^{s} \rangle) \partial_{s} w^{r} dx + \sum_{s=1}^{d} \langle zv^{s} \rangle \int_{\mathbf{T}^{d}} \partial_{s} w^{r} dx = \sum_{s=1}^{d} \int_{\mathbf{T}^{d}} (zv^{s} - \langle zv^{s} \rangle) \partial_{s} w^{r} dx$$

 $(\int_{\mathbf{T}^d} \partial_s w^r dx = 0$ , since this is the integral of a derivative). From (5.25), from the Hölder inequality and (5.23) we get

$$\left| \int_{\mathbf{T}^d} z p^r dx \right| \leqslant \sum_{s=1}^d \|z v^s - \langle z v^s \rangle\|_{L^2} \|\partial_s w^r\|_{L^2} \leqslant K_\omega \|z\|_\omega \sum_{s=1}^d \|v^s\|_1 \|\partial_s w^r\|_{L^2}$$

$$\leqslant K_{\omega} \|z\|_{\omega} \sqrt{\sum_{s=1}^{d} \|v^{s}\|_{1}^{2}} \sqrt{\sum_{s=1}^{d} \|\partial_{s} w^{r}\|_{L^{2}}^{2}} = K_{\omega} \|z\|_{\omega} \|v\|_{1} \sqrt{\sum_{s=1}^{d} \|\partial_{s} w^{r}\|_{L^{2}}^{2}} . \quad (5.26)$$

Now, using (5.26) with (5.20) (5.21) we conclude that  $p^r$  is actually in  $H_0^{-\omega}$ , and  $||p^r||_{-\omega}$  admits the bound (5.19).

Step 3. One has

$$p \in \mathbb{H}_0^{-\omega}$$
,  $||p||_{-\omega} \leqslant K_{\omega} ||v||_1 ||w||_1$  (5.27)

(so, (5.14) holds and the proof is concluded). From Step 2, we know that each component  $p^r$  is in  $H_0^{-\omega}$ ; furthermore, the estimates (5.19) imply

$$||p||_{-\omega}^{2} = \sum_{r=1}^{d} ||p^{r}||_{-\omega}^{2} \leqslant K_{\omega}^{2} ||v||_{1}^{2} \sum_{r,s=1}^{d} ||\partial_{s}w^{r}||_{L^{2}}^{2} = K_{\omega}^{2} ||v||_{1}^{2} ||w||_{1}^{2} , \qquad (5.28)$$

whence the thesis.  $\Box$ 

## 6 Putting things together: an $H^1$ setting for the NS initial value problem, in dimension d=2,3

Let us recall that

$$\mathbf{F}_{+} := \mathbb{H}^{2-\omega}_{\Sigma_{0}}, \quad \mathbf{F} := \mathbb{H}^{1}_{\Sigma_{0}}, \quad \mathbf{F}_{-} := \mathbb{H}^{-\omega}_{\Sigma_{0}}; \qquad \mathcal{A} := \Delta \upharpoonright \mathbb{H}^{2-\omega}_{\Sigma_{0}} : \mathbb{H}^{2-\omega}_{\Sigma_{0}} \to \mathbb{H}^{-\omega}_{\Sigma_{0}}; \quad (6.1)$$

$$\mathcal{P} : \mathbf{F} \times \mathbf{F} \to \mathbf{F}_{-}, \qquad (v, w) \mapsto \mathcal{P}(v, w) := -\mathfrak{L}(v \cdot \partial w).$$

The above Sobolev spaces live, for the moment, on a torus  $\mathbf{T}^d$  of any dimension  $d \in \{2,3,..\}$ ; our previous results about this setting can be summarized as follows. Condition (P1) of Section 2 is fulfilled by the triple  $\mathbf{F}_+, \mathbf{F}, \mathbf{F}_-$ ; according to Proposition 4.1,  $\mathcal{A}$  fulfills conditions (P2-P5) if  $\omega \in (0,1)$ . On the other hand, according to Proposition 5.2,  $\mathcal{P}$  is well defined and fulfills condition (P6) of Section 2 if  $\omega \in (d/2-1,+\infty)$ ; so, the requirements of both Propositions 4.1, 5.2 hold simultaneously if

$$\frac{d}{2} - 1 < \omega < 1 \tag{6.2}$$

Of course, (6.2) can be fulfilled for some  $\omega$  only if d/2 - 1 < 1, i.e., for d = 2 or d = 3; so, (6.2) holds if either

$$d = 2, \qquad \omega \in (0, 1) \tag{6.3}$$

or

$$d = 3, \qquad \omega \in (\frac{1}{2}, 1) \ .$$
 (6.4)

Summing up (and recalling the statements of Propositions 4.1, 5.2 on the function  $\mu$  and the constants B, N, K), we have the following result.

**6.1** Proposition. Let  $d, \omega$  be as in Eqs. (6.3) (6.4). Then, the set  $(\mathbf{F}_+, \mathbf{F}, \mathbf{F}_-, \mathcal{A}, \mathcal{P})$  defined by Eq. (6.1) fulfills all conditions (P1-P6) of Section 2. The function  $\mu$  and the constants B, N, K mentioned in Section 2 can be taken as follows:

$$\mu = \mu_{\omega} \text{ as in (4.4)}, \quad B = 1, \quad N = N_{\omega} \text{ as in (4.6)}, \quad K = K_{\omega} \text{ as in (5.5)}.$$
 (6.5)

The previous proposition allows to apply the full machinery of [9] to the NS initial value problem (3.37). Hereafter we consider, in particular, the condition for global existence presented in [9] and summarized in Proposition 2.1 of the present work.

Global existence for NS with small initial data, when d = 3. Let us keep the definitions (6.1). Global existence for the NS initial value problem (3.37) is well known for any datum in  $\mathbf{F} = \mathbb{H}^1_{\Sigma_0}$ , if d = 2; so, we pass to the case d = 3. Let us recall that, in the three dimensional case, we have the equality (3.29)  $||v||_1 = ||\operatorname{curl} v||_{L^2}$  for all  $v \in \mathbb{H}^1_{\Sigma_0}$ . On the grounds of Propositions 2.1 and 6.1, we can state the following.

**6.2** Proposition. Let d = 3,  $\omega \in (1/2, 1)$  as in (6.4), and  $\mathbf{F}$ , etc., as in (6.1). The solution u of the initial value problem (3.37) is global if the initial datum  $u_0 \in \mathbf{F} = \mathbb{H}^1_{\Sigma_0}$  is such that

$$||u_0||_1 \leqslant \frac{1}{4N_\omega K_\omega} ; \tag{6.6}$$

furthermore, the solution fulfills the bound (of the type (2.14))

$$||u(t)||_1 \leqslant \mathcal{X}(4K_\omega N_\omega ||u_0||_1) e^{-t} ||u_0||_1 \quad \text{for } t \in [0, +\infty) ,$$
 (6.7)

where  $\mathcal{X} \in C([0,1],[1,2])$  is the increasing function defined by (2.15). More roughly, we have

$$||u(t)||_1 \le 2 e^{-t} ||u_0||_1 \quad \text{for } t \in [0, +\infty) .$$
 (6.8)

To get a fully quantitative estimate, let us put

$$\omega := \frac{7}{10} \tag{6.9}$$

(see the forthcoming Remark 6.4 (iii) about this choice). Then, computing numerically the function of t indicated below, we see that

$$\sup_{t \in [0, +\infty)} \int_0^t ds \, e^{-s} \, \mu_{7/10}(t-s) < 1.70 := N_{7/10} \tag{6.10}$$

(in fact the above sup is a maximum, attained at point  $t \in (0.21, 0.22)$ ; for these and other numerical computations one can use, e.g., the MATHEMATICA package). Furthermore, one has (see Appendix D)

$$27.94 < \sup_{k \in \mathbf{Z}_0^d} \mathcal{K}_{7/10}(k) < 32.23 \tag{6.11}$$

(here we are mainly interested in the upper bound 32.23; the lower bound 27.94 is reported only for an appreciation of our uncertainty about  $\sup \mathcal{K}_{7/10}$ ). This implies

$$0.335 < \frac{1}{(2\pi)^{3/2}} \sqrt{\sup_{k \in \mathbf{Z}_0^d} \mathcal{K}_{7/10}(k)} < 0.361 := K_{7/10} . \tag{6.12}$$

The values of  $N_{7/10}$  and  $K_{7/10}$  imply

$$\frac{1}{4N_{7/10}K_{7/10}} > 0.407 ; (6.13)$$

In conclusion, Proposition 6.2 with the choice  $\omega = 7/10$  and the previous evaluations of the related constants yield the following.

**6.3** Corollary. With d=3, the solution u of the initial value problem (3.37) is global if the initial datum  $u_0 \in \mathbf{F} = \mathbb{H}^1_{\Sigma_0}$  is such that

$$||u_0||_1 \leqslant 0.407 \; ; \tag{6.14}$$

furthermore, for all  $t \in [0, +\infty)$ , the solution fulfills the bounds

$$||u(t)||_1 \leqslant \mathcal{X}\left(\frac{||u_0||_1}{0.407}\right) e^{-t} ||u_0||_1 \leqslant 2e^{-t} ||u_0||_1 ,$$
 (6.15)

with  $\mathcal{X}$  as in (2.15).

- **6.4 Remarks.** (i) According to the remark following Proposition 2.1, the global solution u can be constructed by a Picard iteration. A fine analysis of the iteration, based on the the smoothing properties of  $(e^{t\Delta})$ , shows that the function  $(t,x) \mapsto u(t)(x) \equiv u(x,t)$  is in fact  $C^{\infty}$  on  $(0,+\infty) \times \mathbf{T}^3$ , where it satisfies the NS equations in the classical sense; on this point see, e.g., the proof of Theorem 15.2(A) in [4]. Taking into account these facts and the exponential time decay of  $||u(t)||_1$ , we see that u is in  $L^{\infty}([0,+\infty), \mathbb{L}^2_{\Sigma 0}) \cap L^2([0,+\infty), \mathbb{H}^1_{\Sigma 0})$  and fulfills the NS equations in the weak sense; so, u is a weak solution of the NS equations in the sense of [11]. Again by the exponential time decay, it is  $u \in L^4([0,+\infty), \mathbb{H}^1_{\Sigma 0})$ ; this suffices to infer that u is a strong NS solution in the sense of [11] (see the Remark 3.3 on pages 22-23 of this reference).
- (ii) We already mentioned that our criterion (6.14) improves the condition for the existence of a global (strong) solution recently proposed in [10]; in our notations, the condition of the cited paper reads

$$||u_0||_1 \leqslant 0.00724 \tag{6.16}$$

(see page 43 of [10]; 0.00724 is the approximation with three meaningful digits of the quantity indicated therein by  $R_V$ ).

(iii) Of course, in place of  $\omega = 7/10$  one could consider for  $\omega$  other choices in the interval (1/2,1), each one yielding a bound of the type (6.6) for global existence; the best estimate of this type would be obtained maximizing  $1/(4N_{\omega}K_{\omega})$ , for  $\omega$  in (1/2,1). However, a few experiments we did with  $\omega \neq 7/10$  seem to exclude a significant improvement of the bound (6.14).

## 7 Some possible developments.

In [9], we presented the following abstract result:

- 7.1 Proposition. Assume the following:
- (i)  $(\mathbf{F}_+, \mathbf{F}, \mathbf{F}_-, A, \mathcal{P})$  is a set with the properties (P1-P6) of Section 2;  $\mu, B, N, K$  are the function and the constants mentioned therein.
- (ii)  $u_0 \in \mathbf{F}$  is a datum for the initial value problem (2.12).
- (iii)  $u_{ap}$  is an approximate solution of problem (2.12) with domain [0,T) ( $T \in (0,+\infty]$ ), a growth estimator  $\mathcal{D}$  and an error estimator  $\mathcal{E}$ : these expressions indicate three functions  $u_{ap} \in C([0,T), \mathbf{F})$  and  $\mathcal{D}, \mathcal{E} \in C([0,T), [0,+\infty))$  such that

$$||u_{ap}(t)|| \leqslant \mathcal{D}(t) , \qquad (7.1)$$

$$||u_{ap}(t) - e^{t\mathcal{A}}u_0 - \int_0^t ds \ e^{(t-s)\mathcal{A}} \mathcal{P}(u_{ap}(s), u_{ap}(s))|| \leqslant \mathcal{E}(t) \quad for \ t \in [0, T) \ .$$
 (7.2)

(iv) There is a function  $R \in C([0,T),[0,+\infty))$  fulfilling the "control inequality"

$$\mathcal{E}(t) + K \int_0^t ds \, \mu(t-s) e^{-B(t-s)} \Big( 2\mathcal{D}(s) R(s) + R^2(s) \Big) \leqslant R(t) \text{ for } t \in [0, T) . \quad (7.3)$$

Then, problem (2.12) has an (exact) solution  $u \in C([0,T), \mathbf{F})$ , and

$$||u(t) - u_{ap}(t)|| \le R(t) \quad for \ t \in [0, T) \ .$$
 (7.4)

The previous proposition allows to make predictions on the interval of existence of the solution u of (2.12), and on its distance from  $u_{ap}$ , using some information ( $\mathcal{D}$  and  $\mathcal{E}$ ) pertaining to  $u_{ap}$  only; in this sense, we have an estimate for u from an a posteriori analysis of  $u_{ap}$ .

In [9], we presented some applications of this result to the NS equations on  $\mathbf{T}^d$ , taking for  $\mathbf{F}_{\pm}$ ,  $\mathbf{F}$  some Sobolev spaces of sufficiently high order and considering, for example, the Galerkin approximate solutions.

The analysis performed in the present work allows to apply Proposition 7.1 to the NS equations with  $\mathbf{F}_{+} := \mathbb{H}^{2-\omega}_{\Sigma_{0}}, \mathbf{F} := \mathbb{H}^{1}_{\Sigma_{0}}, \mathbf{F}_{-} := \mathbb{H}^{-\omega}_{\Sigma_{0}}$  and  $(d = 2, \omega \in (0, 1) \text{ or})$   $d = 3, \omega \in (1/2, 1)$ .

Our estimates on some related functions and constants (e.g.,  $\mu = \mu_{\omega}$  and  $K = K_{\omega}$  considered in Sections 4-6) allow a fully quantitative implementation of the method. It should be pointed out that the availability of accurate information on  $\mu$ , B, K, etc., is essential for an efficient application of the control inequality (7.3); for example, the time interval [0,T) on which a solution R of (7.3) exists depends sensibly on these data. We plan to illustrate elsewhere some applications of Proposition 7.1 to the NS equations on  $\mathbf{T}^3$ , in the framework of the above spaces  $\mathbf{F}_{\pm}$ ,  $\mathbf{F}$ .

## A Appendix. On the semigroup of the Laplacian

After recalling the definition (3.30)

$$e^{t\Delta}v := \sum_{k \in \mathbf{Z}^d} e^{-t|k|^2} v_k e_k \qquad \forall t \in [0, +\infty), v \in \mathbb{D}'$$

let us fix  $n \in \mathbb{R}$ ,  $\nu \in (0, +\infty)$ . Hereafter, we prove the following result.

**A.1 Proposition.** Let  $t \in (0, +\infty)$ ,  $v \in \mathbb{H}_0^{n-\nu}$ . Then, we have the relations (3.33), (3.34)

$$e^{t\Delta}v \in \mathbb{H}_0^n , \qquad \|e^{t\Delta}v\|_n \leqslant \hat{\mu}_{\nu}(t)e^{-t}\|v\|_{n-\nu} ,$$

$$\hat{\mu}_{\nu}(t) := \begin{cases} \left(\frac{\nu}{2et}\right)^{\frac{\nu}{2}}e^t & for \ 0 < t \leqslant \frac{\nu}{2} ,\\ 1 & for \ t > \frac{\nu}{2} . \end{cases}$$

**Proof.** First of all, since  $v \in \mathbb{D}'_0$  we have  $e^{t\Delta}v \in \mathbb{D}'_0$ . To go on, let us observe that

$$\sum_{k \in \mathbf{Z}_0^d} |k|^{2n} |(e^{t\Delta}v)_k|^2 = \sum_{k \in \mathbf{Z}_0^d} |k|^{2\nu} e^{-2t|k|^2} |k|^{2n-2\nu} |v_k|^2$$
(A.1)

$$\leqslant \left(\sup_{k \in \mathbf{Z}_0^d} |k|^{2\nu} e^{-2t|k|^2}\right) \left(\sum_{k \in \mathbf{Z}_0^d} |k|^{2n-2\nu} |v_k|^2\right) \leqslant \left(\sup_{\vartheta \in [1,+\infty)} U_{\nu t}(\vartheta)\right) \|v\|_{n-\nu}^2 ,$$

$$U_{\nu t}(\vartheta) := \vartheta^{\nu} e^{-2t\vartheta} .$$

An elementary computation gives

$$\sup_{\vartheta \in [1,+\infty)} U_{\nu t}(\vartheta) = \begin{cases} U_{\nu t} \left(\frac{\nu}{2t}\right) = \left(\frac{\nu}{2et}\right)^{\nu} & \text{for } 0 < t \leqslant \frac{\nu}{2} ,\\ U_{\nu t}(1) = e^{-2t} & \text{for } t > \frac{\nu}{2} . \end{cases}$$
(A.2)

Returning to (A.1), we infer that  $e^{t\Delta}v \in \mathbb{H}^n_{\Sigma_0}$  and

$$||e^{t\Delta}v||_n \leqslant \sqrt{\sup_{\vartheta \in [1,+\infty)} U_{\nu t}(\vartheta)} ||v||_{n-\nu} ; \tag{A.3}$$

expressing the above sup via Eq. (A.2) and isolating a factor  $e^{-t}$  we easily get Eqs. (4.3) (4.4).

## B Appendix. Some results on the convolution.

The results we present here are used in the next Appendix to establish some facts about the function  $\mathcal{K}_{\omega}$ , defined on  $\mathbf{Z}_{0}^{d}$  via Eq. (5.2); the cited equation tells us that  $\mathcal{K}_{\omega}(k)$  is a convolutionary sum (apart from a technical detail, i.e., the elimination of 0 and k from the summation domain  $\mathbf{Z}^{d}$ ; we return on this later on).

In this Appendix we give some general results on convolutions, which are stated independently of the subsequent applications to  $\mathcal{K}_{\omega}$ . Of course, the convolution of two functions  $f, g: \mathbf{Z}^d \to [0, +\infty)$  is defined by

$$f * g : \mathbf{Z}^d \to [0, +\infty] , \qquad k \mapsto (f * g)(k) := \sum_{h \in \mathbf{Z}^d} f(h)g(k-h) .$$
 (B.1)

We first consider the case d = 1; our main statement for this case is contained in Proposition B.2, which is preceded by the following definition.

#### B.1 Definition. Consider a function

$$f: \mathbf{Z} \to [0, +\infty] , \qquad k \mapsto f(k) .$$
 (B.2)

f is even if

$$f(-k) = f(k)$$
 for all  $k \in \mathbf{Z}$ ; (B.3)

f is unimodal if it is nondecreasing on  $-\mathbf{N} = \{0, -1, -2, ...\}$ , and nonincreasing on  $\mathbf{N} = \{0, 1, 2, ...\}$ :

$$f(\ell) \leqslant f(m)$$
 for  $\ell, m \in -\mathbf{N}$  and  $\ell \leqslant m$ ,  
 $f(\ell) \geqslant f(m)$  for  $\ell, m \in \mathbf{N}$  and  $\ell \leqslant m$ . (B.4)

Obviously enough, the unimodality of an even f is equivalent to any one of these conditions: f is nonincreasing on  $\mathbb{N}$ , or

$$f(\ell) \geqslant f(m) \quad \text{for } \ell, m \in \mathbf{Z} \text{ and } |\ell| \leqslant |m| .$$
 (B.5)

#### B.2 Proposition. Let us consider two even, unimodal functions

$$p, q: \mathbf{Z} \to [0, +\infty)$$
, (B.6)

and put

$$s := p * q : \mathbf{Z} \to [0, +\infty] , \qquad k \mapsto s(k) = \sum_{h \in \mathbf{Z}} p(h)q(k-h) .$$
 (B.7)

Then, s is itself even and unimodal.

**Proof.** (Adapted from the proof of Proposition 4.5.5 in [5]; the cited result refers to continuous convolutions, with integrals over  $\mathbf{R}$  in place of sums over  $\mathbf{Z}$ ). To prove that s is even we write, for any  $k \in \mathbf{Z}$ :

$$s(-k) = \sum_{h \in \mathbf{Z}} p(h)q(-k-h) = \sum_{j \in \mathbf{Z}} p(-j)q(-k+j) = \sum_{j \in \mathbf{Z}} p(j)q(k-j) = s(k)$$
(B.8)

(the second equality relies on the change of variable h = -j, the third one holds because p, q are even).

Proving the unimodality of s is less trivial. It suffices to prove that s is decreasing on  $\mathbb{N}$ , which can be written as follows:

$$s(\ell) \geqslant s(\ell+1) \quad \text{for } \ell \in \mathbf{N} ;$$
 (B.9)

in the sequel, we fix any  $\ell$  and derive the thesis (B.9). First of all, we write

$$s(\ell) = \sum_{h \in \mathbf{Z}} p(h)q(\ell - h) = \sum_{h = -\infty}^{-1} p(h)q(\ell - h) + \sum_{h = 0}^{\infty} p(h)q(\ell - h) ;$$

now, putting h = -k - 1 in the first sum, h = k in the second one and using p(-k-1) = p(k+1), we get

$$s(\ell) = \sum_{k=0}^{\infty} (p(k+1)q(\ell+k+1) + p(k)q(\ell-k)).$$
 (B.10)

Similarly, writing

$$s(\ell+1) = \sum_{h \in \mathbf{Z}} p(h)q(\ell-h+1) = \sum_{h=-\infty}^{0} p(h)q(\ell-h+1) + \sum_{h=1}^{\infty} p(h)q(\ell-h+1) ,$$

putting h = -k in the first sum, h = k+1 in the second one and using p(-k) = p(k), we get

$$s(\ell+1) = \sum_{k=0}^{\infty} (p(k)q(\ell+k+1) + p(k+1)q(\ell-k)).$$
 (B.11)

Now, consider any  $k \in \mathbb{N}$ . Then  $|k| = k \le k + 1 = |k + 1|$  and  $|\ell - k| \le |\ell| + |k| = \ell + k \le \ell + k + 1 = |\ell + k + 1|$  so that, by Eq. (B.5) with f = q and f = p,

$$p(k) \ge p(k+1)$$
,  $q(\ell-k) \ge q(\ell+k+1)$ . (B.12)

Thus 
$$0 \le [p(k) - p(k+1)][q(\ell-k) - q(\ell+k+1)] = p(k)q(\ell-k) + p(k+1)$$
  
 $q(\ell+k+1) - p(k)q(\ell+k+1) - p(k+1)q(\ell-k)$ , i.e.,

$$p(k+1)q(\ell+k+1) + p(k)q(\ell-k) \ge p(k)q(\ell+k+1) + p(k+1)q(\ell-k)$$
. (B.13)

Now, from (B.10) (B.11) (B.13) we get the thesis (B.9).

Let us extend the previous considerations from the one-dimensional to the d-dimensional case, for arbitrary d.

#### **B.3** Definition. Consider a function

$$f: \mathbf{Z}^d \to [0, +\infty] , \qquad k \mapsto f(k) ;$$
 (B.14)

for each  $r \in \{1, ..., d\}$  and  $k' = (k_1, ..., k_{r-1}, k_{r+1}, ..., k_d) \in \mathbf{Z}^{d-1}$ , put

$$f_{rk'}: \mathbf{Z} \to [0, +\infty], \qquad k_r \mapsto f_{rk'}(k_r) := f(k_1, ..., k_{r-1}, k_r, k_{r+1}, ..., k_d).$$
 (B.15)

We call f even (resp., unimodal) in each variable if, for any  $r \in \{1, ..., d\}$  and  $k' \in \mathbf{Z}^{d-1}$ , the function  $f_{rk'}$  is even (resp., unimodal) in the sense of Definition B.1.

We note that f is even in each variable if and only if

$$f(R_r k) = f(k) \text{ for each } r \in \{1, ..., d\} \text{ and } k \in \mathbf{Z}^d,$$
 (B.16)  
 $R_r k := (k_1, ..., k_{r-1}, -k_r, k_{r+1}, ..., k_d)$ .

If f is even in each variable, recalling Eq. (B.5) we see that the unimodality of f in each variable is equivalent to

$$f(\ell) \geqslant f(m)$$
 if  $\ell, m \in \mathbf{Z}^d$  and  $|\ell_1| \leqslant |m_1|, \dots, |\ell_d| \leqslant |m_d|$ . (B.17)

#### **B.4** Proposition. Let us consider two functions

$$p, q: \mathbf{Z}^d \to [0, +\infty) \tag{B.18}$$

which are even and unimodal in each variable, and put

$$s := p * q : \mathbf{Z}^d \to [0, +\infty] , \qquad k \mapsto s(k) = \sum_{h \in \mathbf{Z}^d} p(h)q(k-h) .$$
 (B.19)

Then, s is itself even and unimodal in each variable.

**Proof.** Let us fix  $r \in \{1, ..., d\}$ ,  $k' = (k_1, ..., k_{r-1}, k_{r+1}, ..., k_d) \in \mathbf{Z}^{d-1}$ , and consider the function  $s_{rk'}: \mathbf{Z} \to [0, +\infty]$ , defined following Eq. (B.15); we must prove that  $s_{rk'}$  is even and unimodal.

To this purpose we note that the definition s := p \* q implies

$$s_{rk'}(k_r) = \sum_{h' \in \mathbf{Z}^{d-1}} (p_{rh'} * q_{r,k'-h'})(k_r)$$
 for each  $k_r \in \mathbf{Z}$ , (B.20)

where the functions  $p_{rh'}, q_{r,k'-h'}: \mathbf{Z} \to [0, +\infty)$  are defined as well following Eq. (B.15). Each of the functions  $p_{rh'}, q_{r,k'-h'}$  is even and unimodal; so, due to Proposition B.2, their convolutions  $p_{rh'} * q_{r,k'-h'}$  are even and unimodal. A sum of even and unimodal functions has the same properties, so our thesis about  $s_{rk'}$  is proved.  $\square$ 

Our last statement about convolutions is obvious, and mentioned only for subsequent citation.

#### **B.5** Definition. A function

$$f: \mathbf{Z}^d \to [0, +\infty] , \qquad k \mapsto f(k)$$
 (B.21)

is symmetric if, for each  $\sigma$  in  $\mathfrak{S}_d$  (the group of permutations of  $\{1,...,d\}$ ) and each  $k = (k_1,...,k_d) \in \mathbf{Z}^d$ , it is

$$f(P_{\sigma}k) = f(k)$$
,  $P_{\sigma}(k) := (k_{\sigma(1)}, ..., k_{\sigma(d)})$ . (B.22)

**B.6** Proposition. If  $p, q : \mathbf{Z}^d \to [0, +\infty)$  are symmetric functions, their convolution p \* q is itself symmetric.

## C Appendix. The function $\mathcal{K}_{\omega}$

Let  $d \in \{2, 3, ...\}$ . In the present Appendix we prove a number of properties of the function  $\mathcal{K}_{\omega}$  on  $\mathbf{Z}_0^d$ , defined by Eq. (5.2); some of these properties were mentioned in Section 5. Some of our results rely on the following two Lemmas.

**C.1** Lemma. Let us consider two radii  $\rho$ ,  $\rho_1$  such that  $2\sqrt{d} \leqslant \rho < \rho_1 \leqslant +\infty$ , and a nonincreasing function  $\chi \in C([\rho - 2\sqrt{d}, \rho_1), [0, +\infty))$ . Then,

$$\sum_{h \in \mathbf{Z}^d, \rho \leqslant |h| < \rho_1} \chi(|h|) \leqslant \frac{2\pi^{d/2}}{\Gamma(d/2)} \int_{\rho - 2\sqrt{d}}^{\rho_1} dt \, (t + \sqrt{d})^{d-1} \chi(t)$$
 (C.1)

(the two sides of the inequality being, possibly,  $+\infty$ ).

**C.2** Lemma. Fix  $\nu \in (d, +\infty)$  and consider, for any  $\lambda \in (0, +\infty)$ , the sum

$$\Delta S_{\nu}(\lambda) := \sum_{h \in \mathbf{Z}^d, |h| \geqslant \lambda + 2\sqrt{d}} \frac{1}{|h|^{\nu}} ; \qquad (C.2)$$

this is finite, and admits the bound

$$\Delta S_{\nu}(\lambda) \leqslant \delta S_{\nu}(\lambda) := \frac{2\pi^{d/2}}{\Gamma(d/2)} \sum_{i=0}^{d-1} \binom{d-1}{i} \frac{d^{d/2-1/2-i/2}}{(\nu-i-1)\lambda^{\nu-i-1}} . \tag{C.3}$$

**Proof.** We apply the previous Lemma, noting that

$$\Delta \mathcal{S}_{\nu}(\lambda) = \sum_{h \in \mathbf{Z}^d, \rho \leqslant |h| < \rho_1} \chi(|h|) , \qquad \rho := \lambda + 2\sqrt{d}, \quad \rho_1 : +\infty, \quad \chi(t) := \frac{1}{t^{\nu}} . \quad (C.4)$$

The inequality (C.1) gives

$$\Delta S_{\nu}(\lambda) \leqslant \frac{2\pi^{d/2}}{\Gamma(d/2)} \int_{\lambda}^{+\infty} dt \, \frac{(t+\sqrt{d})^{d-1}}{t^{\nu}} ; \qquad (C.5)$$

now, writing the expansion  $(t + \sqrt{d})^{d-1} = \sum_{i=0}^{d-1} {d-1 \choose i} t^i d^{d/2-1/2-i/2}$  and integrating term by term, we see that

r.h.s. of (C.5) = 
$$\delta S_{\nu}(\lambda)$$
 . (C.6)

The relations (C.5) and (C.6) give the thesis.

From now on, we make the assumption (5.1)

$$\omega \in (\frac{d}{2} - 1, +\infty) ;$$

as in Eq. (5.2), we consider the function

$$\mathcal{K}_{\omega}: k \in \mathbf{Z}_0^d \mapsto \mathcal{K}_{\omega}(k) := \sum_{h \in \mathbf{Z}_{0k}^d} \frac{1}{|h|^{2\omega}|k-h|^2}.$$

As noted in the comment following (5.2), the sum defining  $\mathcal{K}_{\omega}$  certainly exists in  $(0, +\infty]$ ; hereafter we show its finiteness, with many other properties of the function under investigation.

**C.3 Proposition.** (i) For each  $k \in \mathbf{Z}_0^d$ , it is  $\mathcal{K}_{\omega}(k) < +\infty$ ; furthermore, for any "cutoff"  $\lambda \in (|k|, +\infty)$ , one has

$$\mathcal{K}_{\omega}(k,\lambda) < \mathcal{K}_{\omega}(k) \leqslant \mathcal{K}_{\omega}(k,\lambda) + \delta \mathcal{K}_{\omega}(k,\lambda)$$
 (C.7)

$$\mathcal{K}_{\omega}(k,\lambda) := \sum_{h \in \mathbf{Z}_{0k}^d, |h| < \lambda + 2\sqrt{d}} \frac{1}{|h|^{2\omega}|k-h|^2} , \qquad (C.8)$$

$$\delta \mathcal{K}_{\omega}(k,\lambda) := \frac{2\pi^{d/2}}{\Gamma(d/2)} \sum_{i=0}^{d-1} \binom{d-1}{i} \frac{d^{d/2-1/2-i/2}}{(2\omega+1-i)(\lambda-|k|)^{2\omega+1-i}} . \tag{C.9}$$

(ii) The function  $k \mapsto \mathcal{K}_{\omega}(k)$  is even in each variable and symmetric, i.e.,

$$\mathcal{K}_{\omega}(R_r k) = \mathcal{K}_{\omega}(k) \quad \text{for all } r \in \{1, ..., d\} \text{ and } k \in \mathbf{Z}_0^d ,$$
 (C.10)

$$\mathcal{K}_{\omega}(P_{\sigma}k) = \mathcal{K}_{\omega}(k) \quad \text{for all } \sigma \in \mathfrak{S}_d \text{ and } k \in \mathbf{Z}_0^d$$
 (C.11)

(where, as in Appendix B:  $R_r k := (k_1, ..., -k_r, ..., k_d)$ ,  $\mathfrak{S}_d$  are the permutations of  $\{1, ..., d\}$ , and  $P_{\sigma} k := (k_{\sigma(1)}, ..., k_{\sigma(d)})$ ). Furthermore, one has

$$\mathcal{K}_{\omega}(k) + \frac{1}{|k|^2} + \frac{1}{|k|^{2\omega}} \leqslant \mathcal{K}_{\omega}(\ell) + \frac{1}{|\ell|^2} + \frac{1}{|\ell|^{2\omega}}$$
 (C.12)

if 
$$k, \ell \in \mathbf{Z}_{0k}^d$$
 and  $|k_1| \geqslant |\ell_1|, ..., |k_d| \geqslant |\ell_d|$ .

(iii) It is  $\sup_{k \in \mathbb{Z}_0^d} \mathcal{K}_{\omega}(k) < +\infty$ . For any  $a \in \{1, 2, ...\}$ , we have the bound

$$\sup_{k \in \mathbf{Z}_0^d} \mathcal{K}_{\omega}(k) \leqslant \max(\mathcal{S}_{\omega}(a), \mathcal{R}_{\omega}(a)) , \qquad (C.13)$$

$$S_{\omega}(a) := \max_{k \in I(a)} \mathcal{K}_{\omega}(k) , \qquad I(a) := \{ k \in \mathbf{Z}_0^d \mid 0 \leqslant k_1 \leqslant k_2 \dots \leqslant k_d \leqslant a \} , \quad (C.14)$$

$$\mathcal{R}_{\omega}(a) := \mathcal{K}_{\omega}(0, 0, ..., a+1) + \frac{1}{(a+1)^2} + \frac{1}{(a+1)^{2\omega}} . \tag{C.15}$$

**Proof.** (i) Of course, the bounds (C.7-C.9) to be proved imply finiteness of the sum  $\mathcal{K}_{\omega}(k)$ ; let us derive these bounds, for any fixed  $k \in \mathbf{Z}_0^d$  and cutoff  $\lambda \in (|k|, +\infty)$ . First of all, we write

$$\mathcal{K}_{\omega}(k,\lambda) < \mathcal{K}_{\omega}(k) = \mathcal{K}_{\omega}(k,\lambda) + \Delta \mathcal{K}_{\omega}(k,\lambda) ,$$
 (C.16)

where  $\mathcal{K}_{\omega}(k,\lambda)$  is the finite sum in (C.8), and

$$\Delta \mathcal{K}_{\omega}(k,\lambda) := \sum_{h \in \mathbf{Z}^d, |h| \geqslant \lambda + 2\sqrt{d}} \frac{1}{|h|^{2\omega}|k-h|^2}$$
 (C.17)

(note that  $|h| \ge \lambda + 2\sqrt{d}$  implies  $h \ne 0, k$ ). In principle,  $\Delta \mathcal{K}_{\omega}(k, \lambda) \in (0, +\infty]$ ; hereafter we will prove that

$$\Delta \mathcal{K}_{\omega}(k,\lambda) \leqslant \delta \mathcal{K}_{\omega}(k,\lambda)$$
 (C.18)

where  $\delta \mathcal{K}_{\omega}(k,\lambda) \in (0,+\infty)$  is defined by (C.9); this will give the inequality (C.7) that, with the finiteness of  $\delta \mathcal{K}_{\omega}(k,\lambda)$ , also implies  $\mathcal{K}_{\omega}(k) < +\infty$ . In order to prove (C.18), we put

$$p := \frac{\omega + 1}{\omega} , \qquad q := \omega + 1 ; \tag{C.19}$$

then  $p, q \in (1, +\infty)$  and 1/p + 1/q = 1 so that, by the Hölder inequality,

$$\Delta \mathcal{K}_{\omega}(k,\lambda) \leqslant \left(\sum_{h \in \mathbf{Z}^d, |h| \geqslant \lambda + 2\sqrt{d}} \frac{1}{|h|^{2\omega p}}\right)^{1/p} \left(\sum_{h \in \mathbf{Z}^d, |h| \geqslant \lambda + 2\sqrt{d}} \frac{1}{|k-h|^{2q}}\right)^{1/q}$$

$$= \left(\sum_{h \in \mathbf{Z}^d, |h| \geqslant \lambda + 2\sqrt{d}} \frac{1}{|h|^{2\omega + 2}}\right)^{1/p} \left(\sum_{h \in \mathbf{Z}^d, |h| \geqslant \lambda + 2\sqrt{d}} \frac{1}{|k-h|^{2\omega + 2}}\right)^{1/q}. \quad (C.20)$$

Let us consider the two sums appearing in the last passage of (C.20). A change of variable h' = k - h in the second one gives

$$\sum_{h \in \mathbf{Z}^d, |h| \geqslant \lambda + 2\sqrt{d}} \frac{1}{|k - h|^{2\omega + 2}} = \sum_{h' \in \mathbf{Z}^d, |h' - k| \geqslant \lambda + 2\sqrt{d}} \frac{1}{|h'|^{2\omega + 2}} . \tag{C.21}$$

On the other hand, the inequality  $|h'-k| \ge \lambda + 2\sqrt{d}$  implies  $|h'| = |(h'-k)+k| \ge ||h'-k|-|k|| \ge \lambda + 2\sqrt{d}-|k|$ ; so, the domain of the last sum is contained in the domain  $\{|h'| \ge \lambda + 2\sqrt{d}-|k|\}$ , and we conclude

$$\sum_{h \in \mathbf{Z}^d, |h| \geqslant \lambda + 2\sqrt{d}} \frac{1}{|k - h|^{2\omega + 2}} \leqslant \sum_{h' \in \mathbf{Z}^d, |h'| \geqslant \lambda + 2\sqrt{d} - |k|} \frac{1}{|h'|^{2\omega + 2}} . \tag{C.22}$$

Concerning the first sum in (C.20), it is obvious that

$$\sum_{h \in \mathbf{Z}^d, |h| \geqslant \lambda + 2\sqrt{d}} \frac{1}{|h|^{2\omega + 2}} \leqslant \sum_{h \in \mathbf{Z}^d, |h| \geqslant \lambda + 2\sqrt{d} - |k|} \frac{1}{|h|^{2\omega + 2}} \tag{C.23}$$

(since the right hand side is a sum on a larger domain). We return to (C.20), and insert therein the bounds (C.22) (with h' renamed h) and (C.23); the conclusion is

$$\Delta \mathcal{K}_{\omega}(k,\lambda) \leqslant \left(\sum_{h \in \mathbf{Z}^d, |h| \geqslant \lambda + 2\sqrt{d} - |k|} \frac{1}{|h|^{2\omega + 2}}\right)^{1/p + 1/q}$$

$$= \sum_{h \in \mathbf{Z}^d, |h| \geqslant \lambda + 2\sqrt{d} - |k|} \frac{1}{|h|^{2\omega + 2}} = \Delta S_{2\omega + 2}(\lambda - |k|) \leqslant \delta S_{2\omega + 2}(\lambda - |k|) , \qquad (C.24)$$

where the last two relations follow, respectively, from the definition (C.2) of  $\Delta S_{\nu}(\lambda)$  and from the bound (C.3), here applied with  $\nu \to 2\omega + 2$  and  $\lambda \to \lambda - |k|$  (note that  $2\omega + 2 > d$ , by the assumption (5.1)). On the other hand, explicitating the definition (C.3) of  $\delta S_{\nu}$  we see that

$$\delta S_{2\omega+2}(\lambda - |k|) = \delta \mathcal{K}_{\omega}(k,\lambda) \text{ as in (C.9)};$$
 (C.25)

with (C.24), this yields the thesis (C.18).

(ii) We want to show the properties (C.10-C.12) of  $\mathcal{K}_{\omega}$ . To this purpose, let us define

$$p: \mathbf{Z}^d \to [0, +\infty) , \qquad k \mapsto p(k) := \begin{cases} 1/|k|^{2\omega} & \text{if } k \neq 0, \\ 1 & \text{if } k = 0; \end{cases}$$
 (C.26)

$$q: \mathbf{Z}^d \to [0, +\infty) , \qquad k \mapsto q(k) := \begin{cases} 1/|k|^2 & \text{if } k \neq 0, \\ 1 & \text{if } k = 0. \end{cases}$$
 (C.27)

Then, for  $k \in \mathbf{Z}_0^d$ ,

$$(p*q)(k) = \sum_{h \in \mathbf{Z}^d} p(h)q(k-h) = \sum_{h \in \mathbf{Z}_{0k}^d} p(h)q(k-h) + p(0)q(k) + p(k)q(0) ;$$

explicitating p and q, we can rephrase this as

$$(p*q)(k) = \mathcal{K}_{\omega}(k) + \frac{1}{|k|^2} + \frac{1}{|k|^{2\omega}} \quad \text{for } k \in \mathbf{Z}_0^d .$$
 (C.28)

The functions p, q are even and unimodal in each variable, as well as symmetric (in the sense of Appendix B); by Propositions B.4, B.6, the same properties hold for their convolution  $p * q : \mathbf{Z}^d \to [0, +\infty)$ . So, we have

$$(p*q)(R_rk) = (p*q)(k)$$
 for all  $r \in \{1, ..., d\}$  and  $k \in \mathbf{Z}^d$ , (C.29)

$$(p*q)(P_{\sigma}k) = (p*q)(k)$$
 for all  $\sigma \in \mathfrak{S}_d$  and  $k \in \mathbf{Z}^d$ , (C.30)

$$(p*q)(k) \le (p*q)(\ell) \text{ if } k, \ell \in \mathbf{Z}^d \text{ and } |k_r| \ge |\ell_r| \text{ for } r = 1, ..., d$$
 (C.31)

(the last relation is the inequality (B.17) expressing the unimodality of f := p \* q, with the replacement  $m \to k$ ). For  $k \in \mathbf{Z}_0^d$ , explicitating (p \* q)(k) via Eq. (C.28) (and using the obvious relations  $|R_r k| = |P_{\sigma} k| = |k|$ ) Eqs. (C.29) (C.30) (C.31) yield, respectively, the conclusions (C.10) (C.11) (C.12).

(iii) Let us prove Eq. (C.13) (which, of course, implies the finiteness of  $\sup_{k \in \mathbb{Z}_0^d} \mathcal{K}_{\omega}(k)$ ). To this purpose, we choose any  $a \in \{1, 2, ...\}$  and write

$$\mathbf{Z}_0^d = J(a) \cup L(a) , \qquad (C.32)$$

$$J(a) := \{ k \in \mathbf{Z}_0^d \mid |k_r| \leq a \text{ for all } r \in \{1, ..., d\} \}$$
,

$$L(a) := \{ k \in \mathbf{Z}_0^d \mid |k_s| \geqslant a + 1 \text{ for some } s \in \{1, ..., d\} \}$$
.

Hereafter we will derive the bounds

$$\mathcal{K}_{\omega}(k) \leqslant \mathcal{S}_{\omega}(a)$$
 for all  $k \in J(a)$ , (C.33)

$$\mathcal{K}_{\omega}(k) \leqslant \mathcal{R}_{\omega}(a)$$
 for all  $k \in L(a)$ , (C.34)

yielding the thesis (C.13).

Let  $k \in J(a)$ ; applying a reflection to each negative component of k (if any), and then performing a suitable permutation, we can transform k into an element of the set I(a) in Eq. (C.14); more formally, there is a map  $C = P_{\sigma}R_{r_m}...R_{r_1}$  of  $\mathbf{Z}_0^d$  into itself such that  $Ck \in I(a)$ . Due to the results of (ii),  $\mathcal{K}_{\omega}$  is invariant under C. So,

$$\mathcal{K}_{\omega}(k) = \mathcal{K}_{\omega}(Ck) \leqslant \mathcal{S}_{\omega}(a)$$
,

and (C.33) is proved.

Now, consider any  $k \in L(a)$ . Then  $|k_s| \ge a + 1$  for some  $s \in \{1, ..., d\}$ , which obviously implies

$$|k_r| \ge |\ell_r| \text{ for } r = 1, ..., d, \qquad \ell := (0, ..., \underbrace{a+1}_{\text{place } r}, ..., 0);$$

from here and (C.12) we infer

$$\mathcal{K}_{\omega}(k) + \frac{1}{|k|^2} + \frac{1}{|k|^{2\omega}} \leqslant \mathcal{K}_{\omega}(\ell) + \frac{1}{|\ell|^2} + \frac{1}{|\ell|^{2\omega}}.$$

On the other hand, the definition of  $\ell$  and the symmetry of  $\mathcal{K}_{\omega}$  give  $\mathcal{K}_{\omega}(\ell) + 1/|\ell|^2 + 1/|\ell|^{2\omega} = \mathcal{K}_{\omega}(0,...,0,a+1) + 1/(a+1)^2 + 1/(a+1)^{2\omega} = \mathcal{R}_{\omega}(a)$ , so

$$\mathcal{K}_{\omega}(k) + \frac{1}{|k|^2} + \frac{1}{|k|^{2\omega}} \leqslant \mathcal{R}_{\omega}(a) ;$$

this result is even stronger than the desired relation (C.34).

## D Appendix. Evaluation of $\sup_{k \in \mathbb{Z}_0^3} \mathcal{K}_{7/10}(k)$ .

We specialize the results of the previous Appendix to the case

$$d = 3 , \qquad \omega = \frac{7}{10} ; \qquad (D.1)$$

our aim is to justify the statement (6.11)

$$27.94 < \sup_{k \in \mathbf{Z}_0^d} \mathcal{K}_{7/10}(k) < 32.23 .$$

First of all, let us we write down for the sup of  $\mathcal{K}_{7/10}$  the bound (C.13), with a=1. In this case  $I(a)=I(1)=\{(0,0,1),(0,1,1),(1,1,1)\}$ , so  $\mathcal{S}_{7/10}(1)=\max\left(\mathcal{K}_{7/10}(0,0,1),\mathcal{K}_{7/10}(0,1,1),\mathcal{K}_{7/10}(1,1,1)\right)$ ; furthermore  $\mathcal{R}_{7/10}(1)=\mathcal{K}_{7/10}(0,0,2)+1/4+1/2^{7/5}$ . Eq. (C.13) states that  $\sup_{k\in\mathbf{Z}_0^3}\mathcal{K}_{7/10}\leqslant \max\left(\mathcal{S}_{7/10}(1),\mathcal{R}_{7/10}(1)\right)$ , i.e.,

$$\sup_{k \in \mathbf{Z}_0^3} \mathcal{K}_{7/10}(k) \tag{D.2}$$

$$\leqslant \max\left(\mathcal{K}_{7/10}(0,0,1),\mathcal{K}_{7/10}(0,1,1),\mathcal{K}_{7/10}(1,1,1),\mathcal{K}_{7/10}(0,0,2) + \frac{1}{4} + \frac{1}{2^{7/5}}\right)\,.$$

To evaluate  $\mathcal{K}_{7/10}$  at the points indicated above we use Eqs. (C.7-C.9), with a cutoff  $\lambda = 150$ . The results are:

$$27.94 < \mathcal{K}_{7/10}(0,0,1) < 32.23; \ 27.48 < \mathcal{K}_{7/10}(0,1,1) < 31.77; \ 26.49 < \mathcal{K}_{7/10}(1,1,1) < 30.78;$$

$$25.69 < \mathcal{K}_{7/10}(0,0,2) < 29.98; \quad \mathcal{K}_{7/10}(0,0,2) + \frac{1}{4} + \frac{1}{2^{7/5}} < 30.61 \ . \tag{D.3}$$

(Note that the reminder term  $\delta \mathcal{K}_{7/10}(k,\lambda)$  of Eqs. (C.7-C.9) is  $O(1/\lambda^{2/5})$  for  $\lambda \to +\infty$ ; such a slow decrease at infinity explains why the lower and upper bounds in (D.3) are not very close, even with the fairly large chosen cutoff  $\lambda = 150$ .) From (D.2) and the upper bounds in (D.3) we conclude

$$\sup_{k \in \mathbf{Z}_0^3} \mathcal{K}_{7/10}(k) < 32.23 ,$$

as stated in (6.11). Of course,  $\mathcal{K}_{7/10}(0,0,1) \leqslant \sup_{k \in \mathbf{Z}_0^3} \mathcal{K}_{7/10}(k)$ , and the lower bound for the former in (D.3) implies

$$27.94 < \sup_{k \in \mathbf{Z}_0^3} \mathcal{K}_{7/10}(k) ;$$

our justification of (6.11) is concluded.

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